

EXTERIOR DIFFERENTIAL CALCULUS IN GENERALIZED LIE ALGEBRAS (ALGEBROIDS) CATEGORY WITH APPLICATIONS TO INTERIOR AND EXTERIOR ALGEBRAIC (DIFFERENTIAL) SYSTEMS

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ABSTRACT. A new category of Lie algebras, called generalized Lie algebras, is presented such that classical Lie algebras and Lie-Rinehart algebras are objects of this new category. A new philosophy over generalized Lie algebroids theory is presented using the notion of generalized Lie algebra and examples of objects of the category of generalized Lie algebroids are presented. An exterior differential calculus on generalized Lie algebras is presented and a theorem of Maurer-Cartan type is obtained. Supposing that any submodule (vector subbundle) of a generalized Lie algebra (algebroid) is an interior algebraic (differential) system (IAS (IDS)) for that generalized Lie algebra (algebroid), then the involutivity of the IAS (IDS) in a result of Frobenius type is characterized. Introducing the notion of exterior algebraic (differential) system of a generalized Lie algebra (algebroid), the involutivity of an IAS (IDS) is characterized in a result of Cartan type. Finally, new directions by research in algebraic (differential) symplectic spaces theory are presented.

1. INTRODUCTION

Throughout this paper a ring is a unitary ring and a module over a ring \mathcal{F} is a left module (except for a commutative ring that we consider on a module as a left and right module).

Lie groups and Lie algebras (as linearization of Lie groups) have a vast importance in physics (for example in the classification of elementary particles) [18]. Lie algebras are related to Lie groups via two approaches, first by geometrical bridge and second by fiber bundle theory. Important applications of Lie algebras in physics and mechanics (see [32]) inspired many authors to study these spaces and generalized them to other spaces such as Lie superalgebras (these spaces are important in theoretical physics where they are used to describe the mathematics of supersymmetry) [22], affine (Kac-Moody) Lie algebras (these spaces play an important role in string theory and conformal field theory) [23], quasisimple Lie algebras [14], (locally) extended affine Lie algebras [1, 26, 27] and invariant affine reflection algebras [28].

The basic idea of some branches of mathematics and physics (in particular, noncommutative geometry) is replacing the space M by some algebra of functions on it [10]. Therefore, in mathematical physics, functions and fields are more important than the manifolds on which they are defined [20]. Since the set of vector fields on M , i.e., $\chi(M)$, plays an important role in differential geometry and $\chi(M)$ is the same as the set of all derivations of smooth functions on M , i.e., $Der(\mathcal{F}(M))$, then many authors such as Dubois-Violette generalized the algebra of smooth functions to an arbitrary algebra A and considered

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the Lie algebra $Der(A)$ of all derivations of A as the generalization of the Lie algebra of smooth vector fields [11]. As the exterior differential operator, interior product and Lie derivative are defined on $Der(\mathcal{F}(M))$ and they have basic applications in mathematical physics, then it is important to introduce these derivatives for $Der(A)$. This approach is well-known as "generalization of differential calculus from classical differential geometry to noncommutative geometry" [9, 10, 11].

We know that an classical (usual) algebra over a commutative ring \mathcal{F} is a \mathcal{F} -module A for which there exists a bilinear (biadditive and bihomogenous) operation

$$\begin{array}{ccc} A \times A & \xrightarrow{[\cdot]_A} & A \\ (u, v) & \longmapsto & [u, v]_A \end{array}.$$

In this paper we are going to remove the condition of bihomogeneity and we consider an algebra over a ring (not only commutative) \mathcal{F} as being a \mathcal{F} -module A for which there exists a biadditive operation $[\cdot]_A : A \times A \rightarrow A$. Obviously an classical \mathcal{F} -algebra is an \mathcal{F} -algebra in our direction, but the converse is not true. In the next, we present a Lie algebra over \mathcal{F} as being a \mathcal{F} -algebra $(A, [\cdot]_A)$ such that the biadditive operation $[\cdot]_A$ satisfies

$$LA_1. [u, u]_A = 0, \text{ for any } u \in A,$$

$$LA_2. [u, [v, z]_A]_A + [z, [u, v]_A]_A + [v, [z, u]_A]_A = 0, \text{ for any } u, v, z \in A.$$

Using the Lie \mathcal{F} -algebra $(Der(\mathcal{F}), [\cdot]_{Der(\mathcal{F})})$ of derivations of \mathcal{F} we introduce the notion of generalized Lie \mathcal{F} -algebra as being a \mathcal{F} -module A such that there exists a modules morphism ρ from A to $Der(\mathcal{F})$ and a biadditive operation $[\cdot]_A : A \times A \rightarrow A$, satisfying the following condition

$$[u, fv]_A = f[u, v]_A + \rho(u)(f) \cdot v, \quad \forall u, v \in A \text{ and } f \in \mathcal{F},$$

such that $(A, [\cdot]_A)$ is a Lie \mathcal{F} -algebra.

After presenting our definition of generalized Lie algebra we found that a similar definition had been exhibited by Palais [30] and Rinehart [31], which is called *Lie d-ring* or *Lie-Rinehart algebra* (see [15, 16, 17] for more details). Recall that a pair (A, \mathcal{F}) is called a Lie-Rinehart algebra over R , where R is a commutative and unitary ring, $(\mathcal{F}, [\cdot]_{\mathcal{F}})$ is a commutative classical algebra over R , $(A, [\cdot]_A)$ is a Lie algebra over R and A is the module over \mathcal{F} such that there exists an \mathcal{F} -linear Lie algebras morphism ρ from $(A, [\cdot]_A)$ to $(Der(\mathcal{F}), [\cdot]_{Der(\mathcal{F})})$ satisfies in

$$[u, f \cdot v]_A = f \cdot [u, v]_A + \rho(u)(f) \cdot v, \quad \forall u, v \in A \text{ and } f \in \mathcal{F}.$$

Note that in the Lie-Rinehart algebra, the means of algebra (respectively, Lie algebra) is the classical definition of algebra (respectively, Lie algebra). It is easy to see that a Lie-Rinehart algebra (A, \mathcal{F}) is a generalized Lie algebra over \mathcal{F} (because $(A, [\cdot]_A)$ is a \mathcal{F} -algebra in our approach), but an arbitrary generalized Lie algebra is not necessary a Lie-Rinehart algebra (see Example 2.9).

Using the notion of generalized Lie algebra, in Section 3 we present a new philosophy over theory of generalized Lie algebroids. Some examples of objects of the category of generalized Lie algebroids are presented. Using a similar method used in [5, 6], we propose in Section 4, a new point of view over extension of exterior differential calculus from classical differential geometry to noncommutative geometry using the notion of generalized Lie algebra. In particular, using the locally generalized Lie algebras of an arbitrary generalized Lie algebroid, we obtain an exterior differential calculus for generalized Lie algebroids.

Using the *Cartan's moving frame method*, there exists the following

Theorem (E. Cartan). If $N \in |\mathbf{Man}_n|$ is a Riemannian manifold and $X_\alpha = X_\alpha^i \frac{\partial}{\partial x^i}$, $\alpha \in \overline{1, n}$ is an orthonormal moving frame, then there exists a collection of 1-forms Ω_β^α , $\alpha, \beta \in \overline{1, n}$ uniquely defined by the requirements

$$\Omega_\beta^\alpha = -\Omega_\alpha^\beta$$

and

$$d^F \Theta^\alpha = \Omega_\beta^\alpha \wedge \Theta^\beta, \quad \alpha \in \overline{1, n}$$

where $\{\Theta^\alpha, \alpha \in \overline{1, n}\}$ is the coframe (see [29], p. 151).

It is known that an *r-dimensional distribution on a manifold N* is a mapping \mathcal{D} defined on N , which assigns to each point x of N an r -dimensional linear subspace \mathcal{D}_x of $T_x N$. A vector fields X belongs to \mathcal{D} if we have $X_x \in \mathcal{D}_x$ for each $x \in N$. When this happens we write $X \in \Gamma(\mathcal{D})$. The distribution \mathcal{D} on a manifold N is said to be *differentiable* if for any $x \in N$ there exist r differentiable linearly independent vector fields $X_1, \dots, X_r \in \Gamma(\mathcal{D})$ in a neighborhood of x . The distribution \mathcal{D} is said to be *involutive* if for all vector fields $X, Y \in \Gamma(\mathcal{D})$ we have $[X, Y] \in \Gamma(\mathcal{D})$.

In the classical theory we have the following

Theorem (Frobenius) The distribution \mathcal{D} is involutive if and only if for each $x \in N$ there exists a neighborhood U and $n - r$ linearly independent 1-forms $\Theta^{r+1}, \dots, \Theta^n$ on U which vanish on \mathcal{D} and satisfy the condition

$$d^F \Theta^\alpha = \sum_{\beta \in \overline{r+1, n}} \Omega_\beta^\alpha \wedge \Theta^\beta, \quad \alpha \in \overline{r+1, n},$$

for suitable 1-forms Ω_β^α , $\alpha, \beta \in \overline{r+1, n}$ (see [25], p. 58).

In Section 5, we introduce the definition of an interior algebraic (differential) system (IAS (IDS)) of a generalized Lie algebra (algebroid) and a characterization of the involutivity of an IAS (IDS) in a result of Frobenius type is presented in Theorem 5.9 and Corollary 5.10, respectively. In the classical sense, an exterior differential system (EDS) is a pair (M, \mathcal{I}) consisting of a smooth manifold M and a homogeneous, differentially closed ideal \mathcal{I} in the algebra of smooth differential forms on M (see [7, 13, 19, 24]). Extending the classical notion of EDS to notion of exterior algebraic (differential) system (EAS (EDS)) of a generalized Lie algebra (algebroid), then the involutivity of an IAS (IDS) in a result of Cartan type is presented in the Theorem 5.15 and Corollary 5.16 respectively. Indeed, in this section we show that there exists very close links between EAS (EDS) and the noncommutative geometry of generalized Lie algebras (algebroids). Finally, in Section 6, we present new directions by research in symplectic noncommutative geometry.

2. GENERALIZED LIE ALGEBRAS

In this section, we introduce the generalized Lie \mathcal{F} -algebras category and we present some examples of objects of this category. Also, we obtain some properties of objects of this new category.

Definition 2.1. If A is a \mathcal{F} -module such that there exists an biaditive operation

$$\begin{aligned} A \times A & \xrightarrow{[\cdot]_A} A \\ (u, v) & \longmapsto [u, v]_A \end{aligned} \quad ,$$

then we say that $(A, [\cdot]_A)$ is a \mathcal{F} -algebra or algebra over \mathcal{F} .

If $(A, [,]_A)$ is a \mathcal{F} -algebra such that the operation $[,]_A$ is associative (commutative), then $(A, [,]_A)$ is called an associative (commutative) \mathcal{F} -algebra. Moreover, $(A, [,]_A)$ is called a unitary \mathcal{F} -algebra, if the operation $[,]_A$ has a unitary element.

Remark 2.2. *In the above definition, if \mathcal{F} is a commutative ring and $[,]_A$ is bilinear, then we have the classical definition of algebra over a ring. Thus every classical \mathcal{F} -algebra is an \mathcal{F} -algebra, but the converse is not true. For example if M is a manifold, then $(\chi(M), [,])$, where*

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \quad \forall X, Y \in \chi(M), \quad \forall f \in \mathcal{F}(M),$$

is an $\mathcal{F}(M)$ -algebra but it is not a classical $\mathcal{F}(M)$ -algebra (only a classical \mathbb{R} -algebra, where \mathbb{R} is the field of real numbers).

Definition 2.3. *If \mathcal{F} is a ring, then the set $Der(\mathcal{F})$ of groups morphisms $X : \mathcal{F} \longrightarrow \mathcal{F}$ satisfying the condition*

$$X(f \cdot g) = X(f) \cdot g + f \cdot X(g), \quad \forall f, g \in \mathcal{F},$$

will be called the set of derivations of \mathcal{F} .

If M is a manifold, then it is easy to check that $Der(\mathcal{F}(M)) = \chi(M)$.

Example 2.4. *If we consider the biadditive operation*

$$\begin{array}{ccc} Der(\mathcal{F}) \times Der(\mathcal{F}) & \xrightarrow{[,]_{Der(\mathcal{F})}} & Der(\mathcal{F}) \\ (X, Y) & \longmapsto & [X, Y]_{Der(\mathcal{F})} \end{array},$$

given by

$$[X, Y]_{Der(\mathcal{F})}(f) = X(Y(f)) - Y(X(f)), \quad \forall f \in \mathcal{F},$$

then $(Der(\mathcal{F}), [,]_{Der(\mathcal{F})})$ is an \mathcal{F} -algebra.

Definition 2.5. *If $(A, [,]_A)$ is an \mathcal{F} -algebra such that $[,]_A$ satisfies the conditions:*

LA₁. $[u, u]_A = 0$, for any $u \in A$,

LA₂. $[u, [v, z]_A]_A + [z, [u, v]_A]_A + [v, [z, u]_A]_A = 0$, for any $u, v, z \in A$,

then we will say that $(A, [,]_A)$ is a Lie \mathcal{F} -algebra or Lie algebra over \mathcal{F} .

Example 2.6. *It is easy to check that the \mathcal{F} -algebra $(Der(\mathcal{F}), [,]_{Der(\mathcal{F})})$ is a Lie \mathcal{F} -algebra.*

Using Definition 2.5 we deduce the following:

Proposition 2.7. *If $(A, [,]_A)$ is a Lie \mathcal{F} -algebra, then we have:*

1. $[u, v]_A = -[v, u]_A$, for any $u, v \in A$,
2. $[u, 0]_A = 0$, for any $u \in A$,
3. $[-u, v]_A = -[u, v]_A = [u, -v]_A$, for any $u, v \in A$.

Definition 2.8. *Let A be an \mathcal{F} -module. If there exists a modules morphism ρ from A to $Der(\mathcal{F})$ and a biadditive operation*

$$\begin{array}{ccc} A \times A & \xrightarrow{[,]_A} & A \\ (u, v) & \longmapsto & [u, v]_A \end{array},$$

satisfies in

$$(2.1) \quad [u, fv]_A = f[u, v]_A + \rho(u)(f) \cdot v,$$

for any $u, v \in A$ and $f \in \mathcal{F}$ such that $(A, [\cdot, \cdot]_A)$ is a Lie \mathcal{F} -algebra, then $(A, [\cdot, \cdot]_A, \rho)$ is called a generalized Lie \mathcal{F} -algebra or generalized Lie algebra over \mathcal{F} .

Example 2.9. Let \mathcal{F} be a commutative ring. Obviously it is an \mathcal{F} -module. Now we consider the direct sum

$$\text{Der}(\mathcal{F}) \oplus \mathcal{F} = \{X \oplus f | X \in \text{Der}(\mathcal{F}), f \in \mathcal{F}\}.$$

If we define

$$(X \oplus f) + (Y \oplus g) = (X + Y) \oplus (f + g), \quad h \cdot (X \oplus f) = h \cdot X \oplus h \cdot f,$$

for any $X, Y \in \text{Der}(\mathcal{F})$ and $f, g, h \in \mathcal{F}$, then $\text{Der}(\mathcal{F}) \oplus \mathcal{F}$ is a \mathcal{F} -module. Defining

$$(2.2) \quad [X \oplus f, Y \oplus g]_{\text{Der}(\mathcal{F}) \oplus \mathcal{F}} = [X, Y]_{\text{Der}(\mathcal{F})} \oplus (X(g) - Y(f)),$$

it is easy to see that $[\cdot, \cdot]_{\text{Der}(\mathcal{F}) \oplus \mathcal{F}}$ is biadditive on $\text{Der}(\mathcal{F}) \oplus \mathcal{F}$ and so $(\text{Der}(\mathcal{F}) \oplus \mathcal{F}, [\cdot, \cdot]_{\text{Der}(\mathcal{F}) \oplus \mathcal{F}})$ is an \mathcal{F} -algebra. Direct calculations give us

$$[X \oplus f, X \oplus f]_{\text{Der}(\mathcal{F}) \oplus \mathcal{F}} = 0_{\text{Der}(\mathcal{F})} \oplus 0_{\mathcal{F}} = 0_{\text{Der}(\mathcal{F}) \oplus \mathcal{F}},$$

and

$$\begin{aligned} & [X \oplus f, [Y \oplus g, Z \oplus h]]_{\text{Der}(\mathcal{F}) \oplus \mathcal{F}} + [Y \oplus g, [Z \oplus h, X \oplus f]]_{\text{Der}(\mathcal{F}) \oplus \mathcal{F}} \\ & + [Z \oplus h, [X \oplus f, Y \oplus g]]_{\text{Der}(\mathcal{F}) \oplus \mathcal{F}} = 0_{\text{Der}(\mathcal{F}) \oplus \mathcal{F}}. \end{aligned}$$

Thus $(\text{Der}(\mathcal{F}) \oplus \mathcal{F}, [\cdot, \cdot]_{\text{Der}(\mathcal{F}) \oplus \mathcal{F}})$ is a Lie \mathcal{F} -algebra. Now we define

$$\begin{array}{ccc} \text{Der}(\mathcal{F}) \oplus \mathcal{F} & \xrightarrow{\rho} & \text{Der}(\mathcal{F}) \\ X \oplus f & \longmapsto & X \end{array},$$

for any $X \in \text{Der}(\mathcal{F})$ and $f \in \mathcal{F}$. It is easy to check that ρ is a modules morphism from $\text{Der}(\mathcal{F}) \oplus \mathcal{F}$ to $\text{Der}(\mathcal{F})$. Here, we show that

$$(2.3) \quad [X \oplus f, h \cdot (Y \oplus g)]_{\text{Der}(\mathcal{F}) \oplus \mathcal{F}} = h \cdot [X \oplus f, Y \oplus g]_{\text{Der}(\mathcal{F}) \oplus \mathcal{F}} + \rho(X \oplus f)(h) \cdot (Y \oplus g).$$

Using (2.2) we get

$$\begin{aligned} & [X \oplus f, h \cdot (Y \oplus g)]_{\text{Der}(\mathcal{F}) \oplus \mathcal{F}} = [X \oplus f, h \cdot Y \oplus h \cdot g]_{\text{Der}(\mathcal{F}) \oplus \mathcal{F}} = [X, h \cdot Y]_{\text{Der}(\mathcal{F})} \\ & \oplus (X(h \cdot g) - h \cdot Y(f)) = (h \cdot [X, Y]_{\text{Der}(\mathcal{F})} + X(h) \cdot Y) \oplus (X(h \cdot g) - h \cdot Y(f)) \\ & = h \cdot [X, Y]_{\text{Der}(\mathcal{F})} \oplus h \cdot (X(g) - Y(f)) + X(h) \cdot Y \oplus g \cdot X(h) \\ & = h \cdot ([X, Y]_{\text{Der}(\mathcal{F})} \oplus (X(g) - Y(f))) + X(h) \cdot (Y \oplus g) \\ & = h \cdot [X \oplus f, Y \oplus g]_{\text{Der}(\mathcal{F}) \oplus \mathcal{F}} + \rho(X \oplus f)(h) \cdot (Y \oplus g). \end{aligned}$$

Thus (2.3) holds and consequently $(\text{Der}(\mathcal{F}) \oplus \mathcal{F}, [\cdot, \cdot]_{\text{Der}(\mathcal{F}) \oplus \mathcal{F}}, \rho)$ is a generalized Lie \mathcal{F} -algebra. Relation (2.2) shows that $[\cdot, \cdot]_{\text{Der}(\mathcal{F}) \oplus \mathcal{F}}$ is not \mathcal{F} -bilinear and consequently $(\text{Der}(\mathcal{F}) \oplus \mathcal{F}, [\cdot, \cdot]_{\text{Der}(\mathcal{F}) \oplus \mathcal{F}})$ is not classical Lie algebra over \mathcal{F} . Thus $(\text{Der}(\mathcal{F}) \oplus \mathcal{F}, \mathcal{F})$ is not a Lie-Rinehart algebra over \mathcal{F} .

Proposition 2.10. If $(A, [\cdot, \cdot]_A, \rho)$ is a generalized Lie \mathcal{F} -algebra, then ρ is a Lie algebras morphism from $(A, [\cdot, \cdot]_A)$ to $(\text{Der}(\mathcal{F}), [\cdot, \cdot]_{\text{Der}(\mathcal{F})})$.

Proof. Let $u, v, w \in A$ and $f \in \mathcal{F}$. Since $(A, [,]_A)$ is a Lie \mathcal{F} -algebra, we have the Jacobi identity:

$$(2.4) \quad [[u, v]_A, fw]_A = [u, [v, fw]_A]_A - [v, [u, fw]_A]_A.$$

Using the definition of generalized Lie \mathcal{F} -algebra, we obtain

$$[[u, v]_A, fw]_A = f[[u, v]_A, w]_A + \rho[u, v]_A(f) \cdot w.$$

Similarly, we get

$$\begin{aligned} [u, [v, fw]_A]_A &= [u, f[v, w]_A + \rho(v)(f) \cdot w]_A \\ &= f[u, [v, w]_A]_A + \rho(u)(f) \cdot [v, w]_A \\ &\quad + \rho(v)(f) \cdot [u, w]_A + \rho(u)(\rho(v)(f)) \cdot w, \end{aligned}$$

and

$$\begin{aligned} [v, [u, fw]_A]_A &= [v, f[u, w]_A + \rho(u)(f) \cdot w]_A \\ &= f[v, [u, w]_A]_A + \rho(v)(f) \cdot [u, w]_A \\ &\quad + \rho(u)(f) \cdot [v, w]_A + \rho(v)(\rho(u)(f)) \cdot w. \end{aligned}$$

Setting three above equations in (2.4) and using Jacobi identity, we obtain

$$\begin{aligned} \rho[u, v]_A(f) \cdot w &= \rho(u)(\rho(v)(f)) \cdot w - \rho(v)(\rho(u)(f)) \cdot w \\ &= [\rho(u), \rho(v)]_{Der(\mathcal{F})}(f) \cdot w, \end{aligned}$$

and consequently

$$\rho[u, v]_A = [\rho(u), \rho(v)]_{Der(\mathcal{F})}.$$

Thus modules morphism ρ is a Lie algebras morphism from $(A, [,]_A)$ to $(Der(\mathcal{F}), [,]_{Der(\mathcal{F})})$. □

Definition 2.11. Let $(A', [,]_{A'}, \rho')$ be an another generalized Lie \mathcal{F} -algebra. A generalized Lie \mathcal{F} -algebras morphism from $(A, [,]_A, \rho)$ to $(A', [,]_{A'}, \rho')$ is a Lie \mathcal{F} -algebras morphism φ from $(A, [,]_A)$ to $(A', [,]_{A'})$ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A' \\ \rho \searrow & & \swarrow \rho' \\ & Der(\mathcal{F}) & \end{array}.$$

Thus, we can discuss about the category of generalized Lie \mathcal{F} -algebra.

Remark 2.12. When we consider a basis for A , we let that it is free module and when we consider a basis for $Der(\mathcal{F})$, we let that it is a free module also. Note that, in general, $Der(\mathcal{F})$ is not a free module. For example, if M is an arbitrary manifold then $Der(\mathcal{F}(M)) = \chi(M)$ is not free module. But, if M is a parallelizable manifold, then $Der(\mathcal{F}(M)) = \chi(M)$ is a free module. In particular, if G is a Lie group then $Der(\mathcal{F}(G)) = \chi(G)$ is a free module.

Let $\{\partial_i\}$ be a basis of the Lie \mathcal{F} -algebra of derivations of \mathcal{F} and $\{t_\alpha\}$ be a basis of the generalized Lie \mathcal{F} -algebra $(A, [,]_A, \rho)$. Then we use the notations

$$\rho(t_\alpha) = \rho_\alpha^i \partial_i, \quad [t_\alpha, t_\beta]_A = L_{\alpha\beta}^\gamma t_\gamma, \quad [\partial_i, \partial_j]_{Der(\mathcal{F})} = C_{ij}^k \partial_k,$$

where ρ_α^i , $L_{\alpha\beta}^\gamma$ and C_{ij}^k belong to \mathcal{F} . It is easy to see that $L_{\alpha\beta}^\gamma = -L_{\beta\alpha}^\gamma$ and $C_{ij}^k = -C_{ji}^k$. The components $L_{\alpha\beta}^\gamma$ are called the *structure elements* of the generalized Lie \mathcal{F} -algebra $(A, [,]_A, \rho)$. Using the above proposition, we obtain

$$L_{\alpha\beta}^\gamma \rho_\gamma^k = \rho_\alpha^i \partial_i(\rho_\beta^k) - \rho_\beta^j \partial_j(\rho_\alpha^k) + \rho_\alpha^i \rho_\beta^j C_{ij}^k.$$

Here considering a basis for $Der(\mathcal{F})$ we present a new example of generalized Lie algebra.

Example 2.13. Let \mathcal{F} be a ring such that $Der(\mathcal{F})$ is a free module with basis $\{\partial_i\}_{i \in \overline{1,m}}$. Using the modules morphism $\rho : Der(\mathcal{F}) \rightarrow Der(\mathcal{F})$, we define the operation

$$\begin{aligned} Der(\mathcal{F}) \times Der(\mathcal{F}) &\xrightarrow{\bullet} Der(\mathcal{F}) \\ (X, Y) &\longmapsto X \bullet Y \end{aligned},$$

given by the equality $X \bullet Y = Y^i(X \circ \partial_i) + \rho(X)(Y^i)\partial_i$, where $Y = Y^i\partial_i$ and " \circ " is the usual composition operation. Now, we define

$$(2.5) \quad [X, Y]_{Der(\mathcal{F})}^\bullet = X \bullet Y - Y \bullet X, \quad \forall X, Y \in Der(\mathcal{F}).$$

If $X = X^i\partial_i$ and $Y = Y^j\partial_j$, then we have

$$\begin{aligned} [X, Y]_{Der(\mathcal{F})}^\bullet(fg) &= [X^i\partial_i, Y^j\partial_j]_{Der(\mathcal{F})}^\bullet(fg) = \{(X^i\partial_i) \bullet (Y^j\partial_j) - (Y^j\partial_j) \bullet (X^i\partial_i)\}(fg) \\ &= \{X^iY^j\partial_i \circ \partial_j + X^i\rho(\partial_i)(Y^j)\partial_j - Y^j\rho(\partial_j)(X^i)\partial_i - X^iY^j\partial_j \circ \partial_i\}(fg). \end{aligned}$$

Using the Liebnitz property of ∂_i and ∂_j we get

$$\begin{aligned} [X, Y]_{Der(\mathcal{F})}^\bullet(fg) &= \{X^iY^j(\partial_i\partial_j(f)) + X^i\rho(\partial_i)(Y^j)\partial_j(f) - Y^j\rho(\partial_j)(X^i)\partial_i(f) \\ &\quad - X^iY^j(\partial_j\partial_i(f))\} \cdot g + f \cdot \{X^iY^j(\partial_i\partial_j(g)) + X^i\rho(\partial_i)(Y^j)\partial_j(g) - Y^j\rho(\partial_j)(X^i)\partial_i(g) \\ &\quad - X^iY^j(\partial_j\partial_i(g))\} = [X^i\partial_i, Y^j\partial_j]_{Der(\mathcal{F})}^\bullet(f) \cdot g + f \cdot [X^i\partial_i, Y^j\partial_j]_{Der(\mathcal{F})}^\bullet(g) \\ &= [X, Y]_{Der(\mathcal{F})}^\bullet(f) \cdot g + f \cdot [X, Y]_{Der(\mathcal{F})}^\bullet(g). \end{aligned}$$

Thus $[X, Y]_{Der(\mathcal{F})}^\bullet \in Der(\mathcal{F})$. Moreover, if $f \in \mathcal{F}$, then we have

$$\begin{aligned} [X, fY]_{Der(\mathcal{F})}^\bullet &= [X^i\partial_i, fY^j\partial_j]_{Der(\mathcal{F})}^\bullet = (X^i\partial_i) \bullet (fY^j\partial_j) - (fY^j\partial_j) \bullet (X^i\partial_i) \\ &= fX^iY^j\partial_i \circ \partial_j + X^i\rho(\partial_i)(f)Y^j\partial_j + fX^i\rho(\partial_i)(Y^j)\partial_j - fX^iY^j(\partial_j \circ \partial_i) \\ &\quad - fY^j\rho(\partial_j)(X^i)\partial_i = f[X, Y]_{Der(\mathcal{F})}^\bullet + \rho(X)(f)Y. \end{aligned}$$

Now we check the Jacobi identity for $[\cdot, \cdot]_{Der(\mathcal{F})}$. Using (2.5) we get

$$\begin{aligned} [[X, Y]_{Der(\mathcal{F})}^\bullet, Z]_{Der(\mathcal{F})}^\bullet &= [[X^i\partial_i, Y^j\partial_j]_{Der(\mathcal{F})}^\bullet, Z^k\partial_k]_{Der(\mathcal{F})}^\bullet \\ &= [(X^i\partial_i) \bullet (Y^j\partial_j) - (Y^j\partial_j) \bullet (X^i\partial_i), Z^k\partial_k]_{Der(\mathcal{F})}^\bullet \\ &= [X^iY^j\partial_i \circ \partial_j + X^i\rho(\partial_i)(Y^j)\partial_j - X^iY^j(\partial_j \circ \partial_i) - Y^j\rho(\partial_j)(X^i)\partial_i, Z^k\partial_k]_{Der(\mathcal{F})}^\bullet \\ &= X^iY^jZ^k\partial_i \circ \partial_j \circ \partial_k + X^iY^j\rho(\partial_i)(\rho(\partial_j)(Z^k))\partial_k - Z^k\rho(\partial_k)(X^iY^j)\partial_i \circ \partial_j - X^iY^jZ^k\partial_k \circ \partial_i \circ \partial_j \\ &\quad + X^iZ^k\rho(\partial_i)(Y^j)\partial_j \circ \partial_k + X^i\rho(\partial_i)(Y^j)\rho(\partial_j)(Z^k)\partial_k - X^iZ^k\rho(\partial_i)(Y^j)\partial_k \circ \partial_j - Z^k\rho(\partial_k)(X^i\rho(\partial_i)(Y^j))\partial_j \\ &\quad - X^iY^jZ^k\partial_j \circ \partial_i \circ \partial_k - X^iY^j\rho(\partial_j)(\rho(\partial_i)(Z^k))\partial_k + Z^k\rho(\partial_k)(X^iY^j)\partial_j \circ \partial_i + X^iY^jZ^k\partial_k \circ \partial_j \circ \partial_i \\ &\quad - Y^jZ^k\rho(\partial_j)(X^i)\partial_i \circ \partial_k - Y^j\rho(\partial_j)(X^i)\rho(\partial_i)(Z^k)\partial_k + Y^jZ^k\rho(\partial_j)(X^i)\partial_k \circ \partial_i + Z^k\rho(\partial_k)(Y^j\rho(\partial_j)(X^i))\partial_i. \end{aligned}$$

Also we can obtain similar relations for $[[Y, Z]_{Der(\mathcal{F})}^\bullet, X]_{Der(\mathcal{F})}^\bullet$ and $[[Z, X]_{Der(\mathcal{F})}^\bullet, Y]_{Der(\mathcal{F})}^\bullet$. Using these relations it is easy to see that

$$[[X, Y]_{Der(\mathcal{F})}^\bullet, Z]_{Der(\mathcal{F})}^\bullet + [[Y, Z]_{Der(\mathcal{F})}^\bullet, X]_{Der(\mathcal{F})}^\bullet + [[Z, X]_{Der(\mathcal{F})}^\bullet, Y]_{Der(\mathcal{F})}^\bullet = 0.$$

Thus Jacobi identity holds for this bracket. So, $(Der(\mathcal{F}), [\cdot, \cdot]_{Der(\mathcal{F})}^\bullet, \rho)$ is a new object of the category of generalized Lie algebras

3. GENERALIZED LIE ALGEBROIDS

Let (E, π, M) be an arbitrary vector bundle such that M is paracompact and let $[\mathcal{A}_M]$ be the differentiable structure of the base manifold M , such that for every $(U, \xi_U) \in [\mathcal{A}_M]$, the set U is also the domain of a locally bundle chart.

For any $(U, \xi_U) \in [\mathcal{A}_M]$, it is easily seen that $\mathcal{F}(M)|_U = \{f|_U ; f \in \mathcal{F}(M)\}$ is a unitary ring with respect to the operations

$$\begin{array}{ccc} \mathcal{F}(M)|_U \times \mathcal{F}(M)|_U & \xrightarrow{+} & \mathcal{F}(M)|_U \\ (f|_U, g|_U) & \mapsto & f|_U + g|_U \end{array},$$

and

$$\begin{array}{ccc} \mathcal{F}(M)|_U \times \mathcal{F}(M)|_U & \xrightarrow{\cdot} & \mathcal{F}(M)|_U \\ (f|_U, g|_U) & \mapsto & f|_U \cdot g|_U \end{array}.$$

We remark that any function $f \in \mathcal{F}(M)$ is given by the set of all its restrictions $\{f|_U, (U, \xi_U) \in [\mathcal{A}_M]\}$. It follows easily that $\mathcal{F}(M)$ is a unitary ring with respect to the operations

$$\begin{array}{ccc} \mathcal{F}(M) \times \mathcal{F}(M) & \xrightarrow{+} & \mathcal{F}(M) \\ (f, g) & \mapsto & f + g \end{array},$$

and

$$\begin{array}{ccc} \mathcal{F}(M) \times \mathcal{F}(M) & \xrightarrow{\cdot} & \mathcal{F}(M) \\ (f, g) & \mapsto & f \cdot g \end{array},$$

such that $(f + g)|_U = f|_U + g|_U$ and $(f \cdot g)|_U = f|_U \cdot g|_U$, for any $(U, \xi_U) \in [\mathcal{A}_M]$.

We now consider the restriction vector bundle $(E|_U, \pi, U)$, for any $(U, \xi_U) \in [\mathcal{A}_M]$. We know that $\Gamma(E|_U, \pi, U)$ is a free module over $\mathcal{F}(M)|_U$. For any $(U, \xi_U) \in [\mathcal{A}_M]$, we derive that $\Gamma(E|_U, \pi, U)$ is a $\mathcal{F}(M)|_U$ -module with respect to the operations

$$\begin{array}{ccc} \Gamma(E|_U, \pi, U) \times \Gamma(E|_U, \pi, U) & \xrightarrow{+} & \Gamma(E|_U, \pi, U) \\ (u|_U, v|_U) & \mapsto & u|_U + v|_U \end{array},$$

$$\begin{array}{ccc} \mathcal{F}(M)|_U \times \Gamma(E|_U, \pi, U) & \xrightarrow{\cdot} & \Gamma(E|_U, \pi, U) \\ (f|_U, u|_U) & \mapsto & f|_U \cdot u|_U \end{array}.$$

Any global section $u \in \Gamma(E, \pi, M)$ is given by the set of all its restrictions $\{u|_U, (U, \varphi_U) \in [\mathcal{A}_M]\}$. It is easy to check that $\Gamma(E, \pi, M)$ is a $\mathcal{F}(M)$ -module with respect to the operations

$$\begin{array}{ccc} \Gamma(E, \pi, M) \times \Gamma(E, \pi, M) & \xrightarrow{+} & \Gamma(E, \pi, M) \\ (u, v) & \mapsto & u + v \end{array},$$

and

$$\begin{array}{ccc} \mathcal{F}(M) \times \Gamma(E, \pi, M) & \xrightarrow{\cdot} & \Gamma(E, \pi, M) \\ (f, u) & \mapsto & f \cdot u \end{array},$$

such that $(u + v)|_U = u|_U + v|_U$ and $(f \cdot u)|_U = f|_U \cdot u|_U$, for any $(U, \xi_U) \in [\mathcal{A}_M]$.

Now, let (F, ν, N) be another vector bundle and (φ, φ_0) is a vector bundles morphism from (E, π, M) to (F, ν, N) such that φ_0 is a diffeomorphism from M to N . Using the operation

$$\begin{array}{ccc} \mathcal{F}(M) \times \Gamma(F, \nu, N) & \xrightarrow{\cdot} & \Gamma(F, \nu, N), \\ (f, z) & \mapsto & f \circ \varphi_0^{-1} \cdot z \end{array},$$

we deduce that $\Gamma(F, \nu, N)$ is a $\mathcal{F}(M)$ -module.

Let $(U, \xi_U) \in [\mathcal{A}_M]$ and $(V, \eta_V) \in [\mathcal{A}_N]$ such that $U \subseteq \varphi_0^{-1}(V)$ and let $\{s_a, a \in \overline{1, r}\}$ and $\{t_\alpha, \alpha \in \overline{1, p}\}$ be the local bases for $\Gamma(E|_U, \pi, U)$ and $\Gamma(F_V, \nu, V)$, respectively. Without restriction of generality, we can consider that locally we have the modules morphism

$$\begin{array}{ccc} \Gamma(E|_U, \pi, U) & \xrightarrow{\Gamma(\varphi, \varphi_0)} & \Gamma(F|_V, \nu, V) \\ u|_U = u^a s_a & \longmapsto & \Gamma(\varphi, \varphi_0)(u|_U) \end{array},$$

where $\Gamma(\varphi, \varphi_0)(u|_U) = (u^a \circ \varphi_0^{-1})(\varphi_a^\alpha \circ \varphi_0^{-1})t_\alpha$.

Proposition 3.1. *Let (ρ, η) and (Th, h) be two vector bundles morphisms given by the diagram*

$$\begin{array}{ccccc} F & \xrightarrow{\rho} & TM & \xrightarrow{Th} & TN \\ \downarrow \nu & & \downarrow \tau_M & & \downarrow \tau_N \\ N & \xrightarrow{\eta} & M & \xrightarrow{h} & N \end{array},$$

such that η and h are diffeomorphisms. If $(U, \xi_U) \in [\mathcal{A}_M]$ and $(V, \eta_V) \in [\mathcal{A}_N]$ such that $U \subseteq h^{-1}(V)$, then the modules morphism $\Gamma(Th \circ \rho, h \circ \eta)$ is given locally by the equality

$$(3.1) \quad \Gamma(Th \circ \rho, h \circ \eta)(z^\alpha t_\alpha)(f|_V) = z^\alpha \rho_\alpha^i \left(\frac{\partial(f \circ h|_U)}{\partial x^i} \circ h^{-1} \right)|_V,$$

for any $z^\alpha t_\alpha \in \Gamma(F|_V, \nu, V)$ and $f \in \mathcal{F}(N)$.

Proof. Since $f \in \mathcal{F}(N)$, then $f \circ h \in \mathcal{F}(M)$ and we have

$$\begin{aligned} \frac{\partial f \circ h}{\partial x^i} &= \frac{\partial h^i}{\partial x^i} \cdot \frac{\partial f}{\partial x^i} \circ h \\ &\Downarrow \left\| A_i^i \right\| = \left\| \frac{\partial h^i}{\partial x^i} \right\|^{-1} \\ \frac{\partial f}{\partial x^i} \circ h &= A_i^i \cdot \frac{\partial f \circ h}{\partial x^i} \\ &\Downarrow \\ \frac{\partial f}{\partial x^i} &= (A_i^i \circ h^{-1}) \left(\frac{\partial f \circ h}{\partial x^i} \circ h^{-1} \right). \end{aligned}$$

We now consider that

$$\Gamma(Th \circ \rho, h \circ \eta)(z^\alpha t_\alpha)(f|_V) = (z^\alpha \theta_\alpha^i \frac{\partial f}{\partial x^i}),$$

where the components θ_α^i are given by the equation $\theta_\alpha^i (A_i^i \circ h^{-1}) = \rho_\alpha^i$, which completes the proof. \square

Definition 3.2. *A generalized Lie algebroid is a vector bundle (F, ν, N) given by the diagram:*

$$(3.2) \quad \begin{array}{ccccc} (F, [,]_{F,h}) & \xrightarrow{\rho} & (TM, [,]_{TM}) & \xrightarrow{Th} & (TN, [,]_{TN}) \\ \downarrow \nu & & \downarrow \tau_M & & \downarrow \tau_N \\ N & \xrightarrow{\eta} & M & \xrightarrow{h} & N \end{array},$$

where h and η are arbitrary diffeomorphisms, (ρ, η) is a vector bundles morphism from (F, ν, N) to (TM, τ_M, M) and

$$\begin{array}{ccc} \Gamma(F, \nu, N) \times \Gamma(F, \nu, N) & \xrightarrow{[,]_{F,h}} & \Gamma(F, \nu, N) \\ (u, v) & \longmapsto & [u, v]_{F,h} \end{array},$$

is an operation satisfies in

$$[u, f \cdot v]_{F,h} = f [u, v]_{F,h} + \Gamma(Th \circ \rho, h \circ \eta)(u) f \cdot v, \quad \forall f \in \mathcal{F}(N),$$

such that the 4-tuple $(\Gamma(F, \nu, N), [,]_{F,h})$ is a Lie $\mathcal{F}(N)$ -algebra.

We denote by $((F, \nu, N), [,]_{F,h}, (\rho, \eta))$ the generalized Lie algebroid defined in the above. Moreover, the couple $([,]_{F,h}, (\rho, \eta))$ is called the *generalized Lie algebroid structure*.

Remark 3.3. Note that $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))$ is a generalized Lie algebroid if and only if

$$(\Gamma(F, \nu, N), [\cdot]_{F,h}, \Gamma(Th \circ \rho, h \circ \eta)),$$

is a generalized Lie $\mathcal{F}(N)$ -algebra. This algebra will be called the generalized Lie $\mathcal{F}(N)$ -algebra of the generalized Lie algebroid $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))$.

Proposition 3.4. Let (F, ν, N) be a vector bundle given by the diagram (3.2). If for any $(V, \eta_V) \in [\mathcal{A}_N]$ we have a biadditive operation

$$\begin{array}{ccc} \Gamma(F|_V, \nu, V) \times \Gamma(F|_V, \nu, V) & \xrightarrow{[\cdot]_{F|_V, h}} & \Gamma(F|_V, \nu, V) \\ (u, v) & \longmapsto & [u, v]_{F|_V, h} \end{array},$$

such that the 4-tuple $(\Gamma(F|_V, \nu, V), [\cdot]_{F|_V, h}, \Gamma(Th \circ \rho, h \circ \eta))$ is a generalized Lie $\mathcal{F}(N)|_V$ -algebra, then $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))$ is a generalized Lie algebroid, where $[\cdot]_{F,h}$ is the biadditive operation given by

$$\begin{array}{ccc} \Gamma(F, \nu, N) \times \Gamma(F, \nu, N) & \xrightarrow{[\cdot]_{F,h}} & \Gamma(F, \nu, N) \\ (u, v) & \longmapsto & [u, v]_{F,h} \end{array},$$

such that $[u, v]_{F,h}|_V = [u|_V, v|_V]_{F|_V, h}$, for any $(V, \eta_V) \in [\mathcal{A}_N]$.

Now suppose that $\{t_\alpha\}$ is a local basis for the $\mathcal{F}(N)|_V$ -module of sections of $(F|_V, \nu, V)$ and we put $[t_\alpha, t_\beta]_{F|_V, h} = L_{\alpha\beta}^\gamma t_\gamma$, where $\alpha, \beta, \gamma \in \{1, \dots, p\}$. It is easy to see that $L_{\alpha\beta}^\gamma = -L_{\beta\alpha}^\gamma$. According to Proposition 2.10, $\Gamma(Th \circ \rho, h \circ \eta)$ is a Lie algebras morphism and so we obtain

$$(L_{\alpha\beta}^\gamma \circ h)(\rho_\gamma^k \circ h) = (\rho_\alpha^i \circ h) \frac{\partial(\rho_\beta^k \circ h)}{\partial x^i} - (\rho_\beta^j \circ h) \frac{\partial(\rho_\alpha^k \circ h)}{\partial x^j}.$$

The local real-valued functions $L_{\alpha\beta}^\gamma$ introduced in the above are called the *structure functions* of the generalized Lie algebroid $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))$ (see [2]-[6] for more details).

A morphism from $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))$ to $((F', \nu', N'), [\cdot]_{F',h'}, (\rho', \eta'))$ is a vector bundles morphism (φ, φ_0) from (F, ν, N) to (F', ν', N') such that φ_0 is a diffeomorphism from N to N' and the modules morphism $\Gamma(\varphi, \varphi_0)$ is a Lie $\mathcal{F}(N)$ -algebras morphism from $(\Gamma(F, \nu, N), [\cdot]_{F,h})$ to $(\Gamma(F', \nu', N'), [\cdot]_{F',h'})$. Thus, we can discuss about the category of generalized Lie algebroids.

Here, we present some examples of (generalized) Lie algebroids.

Example 3.5. Any Lie algebroid can be regarded as a generalized Lie algebroid.

Example 3.6. If $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))$ is a generalized Lie algebroid, then we consider the pull-back vector bundle $(h^*F, h^*\nu, M)$. We denote by $(t_\alpha)_{\alpha \in \overline{1, n}}$ the natural base of the generalized Lie $\mathcal{F}(N)|_V$ -algebra $(\Gamma(F|_V, \nu, V), [\cdot]_{F|_V, h}, \Gamma(Th \circ \rho, h \circ \eta))$, where $(V, \eta_V) \in [\mathcal{A}_N]$. Also, we denote by $(\frac{\partial}{\partial x^i})_{i \in \overline{1, m}}$ the natural base of the generalized Lie $\mathcal{F}(M)|_U$ -algebra $(\Gamma(TM|_U, \tau_M, U), [\cdot]_{TM|_U}, \Gamma(Id_{TM}, Id_M))$, where $(U, \xi_U) \in [\mathcal{A}_M]$. Let $(\overset{h^*F}{\rho}, Id_M)$ be the vector bundles morphism from $(h^*F, h^*\nu, M)$ to (TM, τ_M, M) locally given by

$$\begin{array}{ccc} h^*F|_U & \xrightarrow{\overset{h^*F}{\rho}} & TM \\ Z^\alpha T_\alpha(x) & \longmapsto & (Z^\alpha \cdot \rho_\alpha^i \circ h) \frac{\partial}{\partial x^i}(x) \end{array}.$$

We consider the biadditive operation

$$\Gamma(h^*F|_U, h^*\nu, U) \times \Gamma(h^*F|_U, h^*\nu, U) \xrightarrow{[\cdot]_{h^*F|_U}} \Gamma(h^*F|_U, h^*\nu, U),$$

defined by

$$\begin{aligned} [T_\alpha, f|_U \cdot T_\beta]_{h^*F|_U} &= f|_U \cdot (L_{\alpha\beta}^\gamma \circ h)|_U T_\gamma + (\rho_\alpha^i \circ h)|_U \frac{\partial f|_U}{\partial x^i} T_\beta, \\ [f|_U \cdot T_\alpha, T_\beta]_{h^*F|_U} &= -[T_\beta, f|_U \cdot T_\alpha]_{h^*F|_U}, \end{aligned}$$

for any $f \in \mathcal{F}(M)$. After some calculations, we derive that the 4-tuple

$$(\Gamma(h^*F|_U, h^*\nu, U), [\cdot, \cdot]_{h^*F|_U}, \Gamma(\overset{h^*F}{\rho}, Id_M)),$$

is a generalized Lie $\mathcal{F}(M)|_U$ -algebra. So, using the biadditive operation

$$\begin{array}{ccc} \Gamma(h^*F, h^*\nu, M) \times \Gamma(h^*F, h^*\nu, M) & \xrightarrow{[\cdot, \cdot]_{h^*F}} & \Gamma(h^*F, h^*\nu, M) \\ (T, Z) & \mapsto & [T, Z]_{h^*F} \end{array},$$

given by $[T, Z]_{h^*F|_U} = [T|_U, Z|_U]_{h^*F|_U}$, for any $(U, \xi_U) \in [\mathcal{A}_M]$, it results that

$$((h^*F, h^*\nu, M), [\cdot, \cdot]_{h^*F}, (\overset{h^*F}{\rho}, Id_M)),$$

is a Lie algebroid which is called the pull-back Lie algebroid of the generalized Lie algebroid

$$((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta)).$$

Example 3.7. We consider the vector bundle (TN, τ_N, N) given by the diagram:

$$\begin{array}{ccccc} TN & \xrightarrow{T\eta} & TM & \xrightarrow{Th} & (TN, [\cdot, \cdot]_{TN}) \\ \downarrow \tau_N & & \downarrow \tau_M & & \downarrow \tau_N \\ N & \xrightarrow{\eta} & M & \xrightarrow{h} & N \end{array},$$

where M and N are two manifolds and h and η are two diffeomorphisms. We denote by $(t_\alpha)_{\alpha \in \overline{1, n}}$ a base of the $\mathcal{F}(N)|_V$ -module $\Gamma(TN|_V, \tau_N, V)$, where $(V, \eta_V) \in [\mathcal{A}_N]$. Now, we consider the biadditive operation

$$\begin{array}{ccc} \Gamma(TN|_V, \tau_N, V) \times \Gamma(TN|_V, \tau_N, V) & \xrightarrow{[\cdot, \cdot]_{TN|_V, h}} & \Gamma(TN|_V, \tau_N, V) \\ (u, v) & \mapsto & [u, v]_{TN|_V, h} \end{array},$$

defined by

$$\begin{aligned} [t_\alpha, f|_V \cdot t_\beta]_{TN|_V, h} &= \tilde{\theta}([\theta(t_\alpha), \theta(f|_V \cdot t_\beta)]_{TN|_V}), \\ [f|_V \cdot t_\alpha, t_\beta]_{TN|_V, h} &= -[t_\beta, f|_V \cdot t_\alpha]_{TN|_V, h}, \end{aligned}$$

where $\tilde{\theta} := \Gamma(T(h \circ \eta)^{-1}, (h \circ \eta)^{-1})$ and $\theta := \Gamma(Th \circ T\eta, h \circ \eta)$. We denoted by $(\frac{\partial}{\partial x^i})_{i \in \overline{1, n}}$ the natural base for the $\mathcal{F}(M)|_U$ -module $\Gamma(TM|_U, \tau_M, U)$, where $(U, \xi_U) \in [\mathcal{A}_M]$. If $[t_\alpha, t_\beta]_{TN|_V, h} = L_{\alpha\beta}^\gamma t_\gamma$, then

$$L_{\alpha\beta}^\gamma = \tilde{\theta}_j^\gamma (\theta_\alpha^i \frac{\partial \tilde{\theta}_\beta^j}{\partial x^i} - \theta_\beta^i \frac{\partial \tilde{\theta}_\alpha^j}{\partial x^i}), \quad \alpha, \beta, \gamma \in \overline{1, n},$$

where $\tilde{\theta}_\alpha^i$, $i, \alpha \in \overline{1, n}$, are real local functions such that $\theta(t_\alpha) = \theta_\alpha^i \frac{\partial}{\partial x^i}$ and $\tilde{\theta}_j^\gamma$, $j, \gamma \in \overline{1, n}$ are real local functions such that $\tilde{\theta}(\frac{\partial}{\partial x^j}) = \tilde{\theta}_j^\gamma t_\gamma$. Obvious that $\tilde{\theta}(\theta(t_\alpha)) = t_\alpha$ and $\tilde{\theta}_j^\gamma \theta_\gamma^i = \delta_j^i$. For any $u \in \Gamma(TN, \tau_N, N)$, we obtain

$$[u|_V, u|_V]_{TN|_V, h} = \tilde{\theta}([\theta(u|_V), \theta(u|_V)]_{TN|_V}) = \tilde{\theta}(0) = 0.$$

Similarly, for any $u, v \in \Gamma(TN, \tau_N, N)$ and $f \in \mathcal{F}(N)$, we get

$$\begin{aligned} [u|_V, f|_V \cdot v|_V]_{TN|_V, h} &= \tilde{\theta}([\theta(u|_V), \theta(f|_V \cdot v|_V)]_{TN|_V}) = \tilde{\theta}([\theta(u|_V), f|_V \cdot \theta(v|_V)]_{TN|_V}) \\ &= \tilde{\theta}(f|_V \cdot [\theta(u|_V), \theta(v|_V)]_{TN|_V}) + \tilde{\theta}(\theta(u|_V)(f|_V) \cdot \theta(v|_V)) \\ &= f|_V \cdot \tilde{\theta}([\theta(u|_V), \theta(v|_V)]_{TN|_V}) + \theta(u|_V)(f|_V) \cdot \tilde{\theta}(\theta(v|_V)) \\ &= f|_V \cdot [u|_V, v|_V]_{TN|_V, h} + \theta(u|_V)(f|_V) \cdot v|_V. \end{aligned}$$

Relation (3.1) gives us

$$[z|_V, f|_V \cdot v|_V]_{TN|_V, h} = f|_V \cdot [z|_V, v|_V]_{TN|_V, h} + (z^\alpha \rho_\alpha^i (\frac{\partial f \circ h}{\partial x^i} \circ h^{-1})|_V) \cdot v|_V,$$

for any $z, v \in \Gamma(TN, \tau_N, N)$ and $f \in \mathcal{F}(N)$. Also, relation

$$\theta[u|_V, v|_V]_{TN|_V, h} = \theta\{\tilde{\theta}([\theta(u|_V), \theta(v|_V)]_{TN|_V})\} = [\theta(u|_V), \theta(v|_V)]_{TN|_V},$$

implies that

$$[u|_V, [v|_V, z|_V]_{TN|_V, h}]_{TN|_V, h} = \tilde{\theta}([\theta(u|_V), [\theta(v|_V), \theta(z|_V)]_{TN|_V}]_{TN|_V}),$$

for any $u, v, z \in \Gamma(TN, \tau_N, N)$. Now, using the Jacobi identity for the generalized Lie $\mathcal{F}(N)|_V$ -algebra $(\Gamma(TN|_V, \tau_N, V), [,]_{TN|_V})$, we obtain

$$\begin{aligned} &[\theta(u|_V), [\theta(v|_V), \theta(z|_V)]_{TN|_V}]_{TN|_V} + [\theta(v|_V), [\theta(z|_V), \theta(u|_V)]_{TN|_V}]_{TN|_V} \\ &+ [\theta(z|_V), [\theta(u|_V), \theta(v|_V)]_{TN|_V}]_{TN|_V} = 0, \quad \forall u, v, z \in \Gamma(TN, \tau_N, N). \end{aligned}$$

Two last equations give us

$$\begin{aligned} &[u|_V, [v|_V, z|_V]_{TN|_V, h}]_{TN|_V, h} + [z|_V, [u|_V, v|_V]_{TN|_V, h}]_{TN|_V, h} \\ &+ [v|_V, [z|_V, u|_V]_{TN|_V, h}]_{TN|_V, h} = 0, \end{aligned}$$

and so Jacobi identity holds for $[,]_{TN|_V, h}$. Thus, the 4-tuple

$$(\Gamma(TN|_V, \tau_N, V), [,]_{TN|_V, h}, \Gamma(Th \circ T\eta, h \circ \eta)),$$

is a generalized Lie $\mathcal{F}(N)|_V$ -algebra. Therefore, using the biadditive operation

$$\begin{array}{ccc} \Gamma(TN, \nu, N) \times \Gamma(TN, \nu, N) & \xrightarrow{[\cdot]_{TN, h}} & \Gamma(TN, \nu, N) \\ (u, v) & \longmapsto & [u, v]_{TN, h} \end{array},$$

given by $[u, v]_{TN, h}|_V = [u|_V, v|_V]_{TN|_V, h}$, for any $(V, \eta_V) \in [\mathcal{A}_N]$, it results that

$$((TN, \tau_N, N), [,]_{TN, h}, (T\eta, \eta)),$$

is a generalized Lie algebroid.

For any diffeomorphisms η and h , new and interesting generalized Lie algebroids structures for the tangent vector bundle (TN, τ_N, N) are obtained. In particular, using arbitrary isometries (symmetries, translations, rotations,...) for the Euclidean 3-dimensional space Σ , and arbitrary basis for the module of sections we obtain a lot of generalized Lie algebroids structures for the tangent vector bundle $(T\Sigma, \tau_\Sigma, \Sigma)$.

Example 3.8. Let $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$ be a Lie algebroid, $(U, \xi_U) \in [\mathcal{A}_N]$ and $(V, \eta_V) \in [\mathcal{A}_N]$ such that $U \subseteq h^{-1}(V)$. We denote by $(t_\alpha)_{\alpha \in \overline{1, n}}$ the natural base of the generalized Lie $\mathcal{F}(N)_{|V}$ -algebra $(\Gamma(F_{|V}, \nu, V), [\cdot, \cdot]_{F_{|V}}, \Gamma(\rho, Id_N))$. If $h \in Diff(N)$, then we consider the biadditive operation

$$\begin{aligned} \Gamma(F_{|V}, \nu, V) \times \Gamma(F_{|V}, \nu, V) &\xrightarrow{[\cdot, \cdot]_{F_{|V}}, h} \Gamma(F_{|V}, \nu, V) \\ (u_{|V}, v_{|V}) &\longmapsto [u_{|V}, v_{|V}]_{F_{|V}, h}, \end{aligned}$$

defined by

$$(3.3) \quad [t_\alpha, f_{|V} \cdot t_\beta]_{F_{|V}, h} = f_{|V} \cdot [t_\alpha, t_\beta]_{F_{|V}} + \rho_\alpha^i \left(\frac{\partial f \circ h}{\partial x^i} \circ h^{-1} \right)_{|V} \cdot t_\beta,$$

and

$$[f_{|V} \cdot t_\alpha, t_\beta]_{F_{|V}, h} = -[t_\beta, f_{|V} \cdot t_\alpha]_{F_{|V}, h},$$

for any $f \in \mathcal{F}(N)$. Easily, we obtain $[f_{|V} \cdot t_\alpha, f_{|V} \cdot t_\alpha]_{F_{|V}, h} = 0$ and consequently

$$[u_{|V}, u_{|V}]_{F_{|V}, h} = 0, \quad \forall u \in \Gamma(F, \nu, N).$$

Since $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$ is a Lie algebroid, then

$$[t_\alpha, [t_\beta, t_\gamma]_{F_{|V}}]_{F_{|V}} + [t_\gamma, [t_\alpha, t_\beta]_{F_{|V}}]_{F_{|V}} + [t_\beta, [t_\gamma, t_\alpha]_{F_{|V}}]_{F_{|V}} = 0,$$

which gives us

$$(3.4) \quad \begin{aligned} &L_{\beta\gamma}^\theta L_{\alpha\theta}^\mu + \rho_\alpha^i \left(\frac{\partial L_{\beta\gamma}^\theta}{\partial x^i} \circ h \right)_{|V} + L_{\alpha\beta}^\theta L_{\gamma\theta}^\mu + \rho_\gamma^i \left(\frac{\partial L_{\alpha\beta}^\theta}{\partial x^i} \circ h \right)_{|V} \\ &+ L_{\gamma\alpha}^\theta L_{\beta\theta}^\mu + \rho_\beta^i \left(\frac{\partial L_{\gamma\alpha}^\theta}{\partial x^i} \circ h \right)_{|V} = 0, \end{aligned}$$

where ρ_α^i and $L_{\alpha\beta}^\theta$ are real-valued functions defined on U . As

$$\begin{aligned} &\Gamma(Th \circ \rho, h)[t_\alpha, t_\beta]_{F_{|V}} = \Gamma(Th, h)[\Gamma(\rho, Id_N)t_\alpha, \Gamma(\rho, Id_N)t_\beta]_{TN_{|V}} \\ &= [\Gamma(Th, h) \circ \Gamma(\rho, Id_N)t_\alpha, \Gamma(Th, h) \circ \Gamma(\rho, Id_N)t_\beta]_{TN_{|V}} \\ &= [\Gamma(Th \circ \rho, h)t_\alpha, \Gamma(Th \circ \rho, h)t_\beta]_{TN_{|V}}, \end{aligned}$$

we get

$$(3.5) \quad \begin{aligned} &L_{\alpha\beta}^\gamma \rho_\gamma^k \left(\frac{\partial h^i}{\partial x^k} \circ h^{-1} \right)_{|V} \frac{\partial}{\partial x^i} = (\tilde{\theta}_\alpha^j \frac{\partial \tilde{\theta}_\beta^i}{\partial x^j} - \tilde{\theta}_\beta^j \frac{\partial \tilde{\theta}_\alpha^i}{\partial x^j}) \frac{\partial}{\partial x^i} \\ &\quad \Updownarrow \\ &L_{\alpha\beta}^\gamma \rho_\gamma^k \left(\frac{\partial h^i}{\partial x^k} \circ h^{-1} \right)_{|V} \frac{\partial}{\partial x^i} = (\tilde{\theta}_\alpha^j \frac{\partial [\rho_\beta^k (\frac{\partial h^i}{\partial x^k} \circ h^{-1})_{|V}]}{\partial x^j} - \tilde{\theta}_\beta^j \frac{\partial [\rho_\alpha^k (\frac{\partial h^i}{\partial x^k} \circ h^{-1})_{|V}]}{\partial x^j}) \frac{\partial}{\partial x^i} \\ &\quad \Updownarrow \\ &L_{\alpha\beta}^\gamma \rho_\gamma^k \left(\frac{\partial h^i}{\partial x^k} \circ h^{-1} \right)_{|V} \frac{\partial}{\partial x^i} = (\tilde{\theta}_\alpha^j \frac{\partial \rho_\beta^k}{\partial x^j} - \tilde{\theta}_\beta^j \frac{\partial \rho_\alpha^k}{\partial x^j}) \left(\frac{\partial h^i}{\partial x^k} \circ h^{-1} \right)_{|V} \frac{\partial}{\partial x^i} \\ &\quad \Updownarrow \\ &L_{\alpha\beta}^\gamma \rho_\gamma^k = \tilde{\theta}_\alpha^j \frac{\partial \rho_\beta^k}{\partial x^j} - \tilde{\theta}_\beta^j \frac{\partial \rho_\alpha^k}{\partial x^j} \\ &\quad \Updownarrow \\ &L_{\alpha\beta}^\gamma \rho_\gamma^k = \rho_\alpha^j \left(\frac{\partial \rho_\beta^k \circ h}{\partial x^j} \circ h^{-1} \right)_{|V} - \rho_\beta^j \left(\frac{\partial \rho_\alpha^k \circ h}{\partial x^j} \circ h^{-1} \right)_{|V} \\ &\quad \Updownarrow \\ &L_{\alpha\beta}^\gamma \rho_\gamma^k \circ h = (\rho_\alpha^j \circ h) \frac{\partial \rho_\beta^k \circ h}{\partial x^j} - (\rho_\beta^j \circ h) \frac{\partial \rho_\alpha^k \circ h}{\partial x^j}. \end{aligned}$$

Using (3.3) we get

$$\begin{aligned}
[t_\alpha, [t_\beta, f|_V t_\gamma]_{F|V, h}]_{F|V, h} &= f|_V L_{\beta\gamma}^\theta L_{\alpha\theta}^\mu t_\mu + \rho_\alpha^i \left(\frac{\partial f \circ h}{\partial x^i} \circ h^{-1} \right) |_V L_{\beta\gamma}^\theta t_\theta \\
&+ f|_V \rho_\alpha^i \left(\frac{\partial L_{\beta\gamma}^\theta \circ h}{\partial x^i} \circ h^{-1} \right) |_V t_\theta + \rho_\beta^j \left(\frac{\partial f \circ h}{\partial x^j} \circ h^{-1} \right) |_V L_{\alpha\gamma}^\theta t_\theta \\
&+ \rho_\alpha^i \left(\frac{\partial \rho_\beta^j \circ h}{\partial x^i} \circ h^{-1} \right) |_V \left(\frac{\partial f \circ h}{\partial x^j} \circ h^{-1} \right) |_V t_\gamma + \rho_\beta^j \rho_\alpha^i \frac{\partial}{\partial x^i} \left(\frac{\partial f \circ h}{\partial x^j} \circ h^{-1} \right) |_V t_\gamma, \\
[f|_V t_\gamma, [t_\alpha, t_\beta]_{F|V, h}]_{F|V, h} &= f|_V L_{\alpha\beta}^\theta L_{\gamma\theta}^\mu t_\mu - L_{\alpha\beta}^\theta \rho_\theta^i \left(\frac{\partial f \circ h}{\partial x^i} \circ h^{-1} \right) |_V t_\gamma \\
&+ f|_V \rho_\gamma^i \left(\frac{\partial L_{\alpha\beta}^\theta \circ h}{\partial x^i} \circ h^{-1} \right) |_V t_\theta,
\end{aligned}$$

and

$$\begin{aligned}
[t_\beta, [f|_V t_\gamma, t_\alpha]_{F|V, h}]_{F|V, h} &= f|_V L_{\gamma\alpha}^\theta L_{\beta\theta}^\mu t_\mu - \rho_\beta^i \left(\frac{\partial f \circ h}{\partial x^i} \circ h^{-1} \right) |_V L_{\alpha\gamma}^\theta t_\theta \\
&- \rho_\alpha^j \left(\frac{\partial f \circ h}{\partial x^j} \circ h^{-1} \right) |_V L_{\beta\gamma}^\theta t_\theta - \rho_\beta^i \left(\frac{\partial \rho_\alpha^j \circ h}{\partial x^i} \circ h^{-1} \right) |_V \left(\frac{\partial f \circ h}{\partial x^j} \circ h^{-1} \right) |_V t_\gamma \\
&- \rho_\alpha^i \rho_\beta^j \frac{\partial}{\partial x^j} \left(\frac{\partial f \circ h}{\partial x^i} \circ h^{-1} \right) |_V t_\gamma + f|_V \rho_\beta^i \left(\frac{\partial L_{\gamma\alpha}^\theta \circ h}{\partial x^i} \circ h^{-1} \right) |_V t_\theta.
\end{aligned}$$

Summing above three equations and using (3.4) and (3.5) imply that

$$\begin{aligned}
&[t_\alpha, [t_\beta, f|_V t_\gamma]_{F|V, h}]_{F|V, h} + [f|_V t_\gamma, [t_\alpha, t_\beta]_{F|V, h}]_{F|V, h} \\
&+ [t_\beta, [f|_V t_\gamma, t_\alpha]_{F|V, h}]_{F|V, h} = 0.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&[u|_V, [v|_V, z|_V]_{F|V, h}]_{F|V, h} + [z|_V, [u|_V, v|_V]_{F|V, h}]_{F|V, h} \\
&+ [v|_V, [z|_V, u|_V]_{F|V, h}]_{F|V, h} = 0,
\end{aligned}$$

for any $u, v, z \in \Gamma(F, \nu, N)$. Therefore, the 4-tuple $(\Gamma(F|_V, \nu, V), [,]_{F|V, h})$ is a generalized Lie $\mathcal{F}(N)|_V$ -algebra. Moreover, using the biadditive operation given by

$$\begin{aligned}
\Gamma(F, \nu, N) \times \Gamma(F, \nu, N) &\xrightarrow{[\cdot]_{F, h}} \Gamma(F, \nu, N) \\
(u, v) &\longmapsto [u, v]_{F, h},
\end{aligned}$$

given by $[u, v]_{F, h}|_V = [u|_V, v|_V]_{F|V, h}$, for any $(V, \eta_V) \in [\mathcal{A}_N]$, we derive that $((F, \nu, N), [,]_{F, h}, (T\eta \circ \rho, \eta))$ is a generalized Lie algebroid.

We remark that for any Lie algebroid $((F, \nu, N), [,]_F, (\rho, Id_N))$ and for any $h \in Diff(N)$ we obtain a new generalized Lie algebroid $((F, \nu, N), [,]_{F, h}, (\rho, Id_N))$.

In the future we will use the same notations for the global Lie bracket and local Lie bracket, but we understand the difference with respect to the context.

4. EXTERIOR DIFFERENTIAL CALCULUS

In the next we consider \mathcal{F} as a commutative ring. In this section, we propose an exterior differential calculus in the general framework of a generalized Lie algebras. Note that for any $q \in \mathbb{N}^*$ we denote by (Σ_q, \circ) the permutations group of the set $\{1, 2, \dots, q\}$.

Definition 4.1. Let $(A, [,]_A, \rho)$ be a generalized Lie algebra. A q -linear operation

$$\begin{array}{ccc} A^q & \xrightarrow{\omega} & \mathcal{F} \\ (z_1, \dots, z_q) & \longmapsto & \omega(z_1, \dots, z_q) \end{array},$$

such that

$$\omega(z_{\sigma(1)}, \dots, z_{\sigma(q)}) = \text{sgn}(\sigma) \omega(z_1, \dots, z_q),$$

for any $z_1, \dots, z_q \in A$ and for any $\sigma \in \Sigma_q$, will be called a form of degree q (or q -form) and the set of q -forms will be denoted by $\Lambda^q(A)$.

Using the above definition, if $\omega \in \Lambda^q(A)$, then $\omega(z_1, \dots, z, \dots, z, \dots, z_q) = 0$. Therefore, if $\omega \in \Lambda^q(A)$, then

$$\omega(z_1, \dots, z_i, \dots, z_j, \dots, z_q) = -\omega(z_1, \dots, z_j, \dots, z_i, \dots, z_q).$$

Proposition 4.2. If $q \in \mathbb{N}$, then $\Lambda^q(A)$ is an \mathcal{F} -module.

Definition 4.3. If $\omega \in \Lambda^q(A)$ and $\theta \in \Lambda^r(A)$, then the $(q+r)$ -form $\omega \wedge \theta$ defined by

$$\begin{aligned} \omega \wedge \theta(z_1, \dots, z_{q+r}) &= \sum_{\substack{\sigma(1) < \dots < \sigma(q) \\ \sigma(q+1) < \dots < \sigma(q+r)}} \text{sgn}(\sigma) \omega(z_{\sigma(1)}, \dots, z_{\sigma(q)}) \cdot \theta(z_{\sigma(q+1)}, \dots, z_{\sigma(q+r)}) \\ (4.1) \quad &= \frac{1}{q!r!} \sum_{\sigma \in \Sigma_{q+r}} \text{sgn}(\sigma) \omega(z_{\sigma(1)}, \dots, z_{\sigma(q)}) \cdot \theta(z_{\sigma(q+1)}, \dots, z_{\sigma(q+r)}), \end{aligned}$$

for any $z_1, \dots, z_{q+r} \in A$, will be called the exterior product of the forms ω and θ .

Using the above definition, we obtain

Theorem 4.4. Let $f \in \mathcal{F}$, $\omega, \bar{\omega} \in \Lambda^q(A)$, $\theta, \bar{\theta} \in \Lambda^r(A)$ and $\eta \in \Lambda^s(A)$. Then we have

$$\begin{aligned} \omega \wedge \theta &= (-1)^{q \cdot r} \theta \wedge \omega, \quad (f \cdot \omega) \wedge \theta = f \cdot (\omega \wedge \theta) = \omega \wedge (f \cdot \theta), \\ (\omega + \bar{\omega}) \wedge \theta &= \omega \wedge \theta + \bar{\omega} \wedge \theta, \quad \omega \wedge (\theta + \bar{\theta}) = \omega \wedge \theta + \omega \wedge \bar{\theta}, \\ (\omega \wedge \theta) \wedge \eta &= \omega \wedge (\theta \wedge \eta). \end{aligned}$$

Theorem 4.5. If we set

$$\Lambda(A) = \bigoplus_{q \geq 0} \Lambda^q(A),$$

then $(\Lambda(A), \wedge)$ is an associative classical \mathcal{F} -algebra. This algebra will be called the exterior algebra of the \mathcal{F} -algebra $(A, [,]_A)$.

If $\{t^\alpha, \alpha \in \overline{1, p}\}$ is the cobase associated to the base $\{t_\alpha, \alpha \in \overline{1, p}\}$ of the \mathcal{F} -algebra $(A, [,]_A)$, then for any $q \in \overline{1, p}$ we define C_p^q exterior forms of the type $t^{\alpha_1} \wedge \dots \wedge t^{\alpha_q}$, such that $1 \leq \alpha_1 < \dots < \alpha_q \leq p$. The set

$$\{t^{\alpha_1} \wedge \dots \wedge t^{\alpha_q} | 1 \leq \alpha_1 < \dots < \alpha_q \leq p\},$$

is a basis for the \mathcal{F} -module $\Lambda^q(A)$. Therefore, if $\omega \in \Lambda^q(A)$, then

$$\omega = \omega_{\alpha_1 \dots \alpha_q} t^{\alpha_1} \wedge \dots \wedge t^{\alpha_q}.$$

In particular, if ω is an exterior p -form, then it can be written as

$$\omega = f \cdot t^1 \wedge \dots \wedge t^p,$$

where $f \in \mathcal{F}$.

Definition 4.6. For any $z \in A$, the operator

$$\Lambda(A) \xrightarrow{L_z} \Lambda(A) ,$$

defined by

$$L_z(f) = \rho(z)(f), \quad \forall f \in \mathcal{F},$$

and

$$(4.2) \quad L_z \omega(z_1, \dots, z_q) = \rho(z)(\omega(z_1, \dots, z_q)) - \sum_{i=1}^q \omega((z_1, \dots, [z, z_i]_A, \dots, z_q)),$$

for any $\omega \in \Lambda^q(A)$ and $z_1, \dots, z_q \in A$, will be called the covariant Lie derivative with respect to the element z .

Theorem 4.7. If $z \in A$, $\omega \in \Lambda^q(A)$ and $\theta \in \Lambda^r(A)$, then

$$L_z(\omega \wedge \theta) = L_z \omega \wedge \theta + \omega \wedge L_z \theta.$$

Proof. Let $z_1, \dots, z_{q+r} \in A$. (4.1) and (4.2) give us

$$\begin{aligned} L_z(\omega \wedge \theta)(z_1, \dots, z_{q+r}) &= \rho(z)((\omega \wedge \theta)(z_1, \dots, z_{q+r})) \\ &- \sum_{i=1}^{q+r} (\omega \wedge \theta)((z_1, \dots, [z, z_i]_A, \dots, z_{q+r})) \\ &= \rho(z) \left(\sum_{\substack{\sigma(1) < \dots < \sigma(q) \\ \sigma(q+1) < \dots < \sigma(q+r)}} \text{sgn}(\sigma) \omega(z_{\sigma(1)}, \dots, z_{\sigma(q)}) \cdot \theta(z_{\sigma(q+1)}, \dots, z_{\sigma(q+r)}) \right) \\ &- \sum_{i=1}^{q+r} (\omega \wedge \theta)((z_1, \dots, [z, z_i]_A, \dots, z_{q+r})). \end{aligned}$$

Therefore we get

$$\begin{aligned} L_z(\omega \wedge \theta)(z_1, \dots, z_{q+r}) &= \sum_{\substack{\sigma(1) < \dots < \sigma(q) \\ \sigma(q+1) < \dots < \sigma(q+r)}} \text{sgn}(\sigma) \rho(z)(\omega(z_{\sigma(1)}, \dots, z_{\sigma(q)})) \\ &\cdot \theta(z_{\sigma(q+1)}, \dots, z_{\sigma(q+r)}) + \sum_{\substack{\sigma(1) < \dots < \sigma(q) \\ \sigma(q+1) < \dots < \sigma(q+r)}} \text{sgn}(\sigma) \omega(z_{\sigma(1)}, \dots, z_{\sigma(q)}) \\ &\cdot \rho(z)(\theta(z_{\sigma(q+1)}, \dots, z_{\sigma(q+r)})) - \sum_{\substack{\sigma(1) < \dots < \sigma(q) \\ \sigma(q+1) < \dots < \sigma(q+r)}} \text{sgn}(\sigma) \\ &\sum_{i=1}^q \omega(z_{\sigma(1)}, \dots, [z, z_{\sigma(i)}]_A, \dots, z_{\sigma(q)}) \cdot \theta(z_{\sigma(q+1)}, \dots, z_{\sigma(q+r)}) \\ &- \sum_{\substack{\sigma(1) < \dots < \sigma(q) \\ \sigma(q+1) < \dots < \sigma(q+r)}} \text{sgn}(\sigma) \sum_{i=q+1}^{q+r} \omega(z_{\sigma(1)}, \dots, z_{\sigma(q)}) \\ &\cdot \theta(z_{\sigma(q+1)}, \dots, [z, z_{\sigma(i)}]_A, \dots, z_{\sigma(q+r)}), \end{aligned}$$

and consequently

$$\begin{aligned}
 L_z(\omega \wedge \theta)(z_1, \dots, z_{q+r}) &= \sum_{\substack{\sigma(1) < \dots < \sigma(q) \\ \sigma(q+1) < \dots < \sigma(q+r)}} sgn(\sigma) L_z \omega(z_{\sigma(1)}, \dots, z_{\sigma(q)}) \\
 &\quad \cdot \theta(z_{\sigma(q+1)}, \dots, z_{\sigma(q+r)}) + \sum_{\substack{\sigma(1) < \dots < \sigma(q) \\ \sigma(q+1) < \dots < \sigma(q+r)}} sgn(\sigma) \sum_{i=q+1}^{q+r} \omega(z_{\sigma(1)}, \dots, z_{\sigma(q)}) \\
 &\quad \cdot L_z \theta(z_{\sigma(q+1)}, \dots, z_{\sigma(q+r)}) \\
 &= (L_z \omega \wedge \theta + \omega \wedge L_z \theta)(z_1, \dots, z_{q+r}).
 \end{aligned}$$

□

Definition 4.8. For any $z \in A$, the operator

$$\begin{aligned}
 \Lambda(A) &\xrightarrow{i_z} \Lambda(A) \\
 \Lambda^q(A) \ni \omega &\longmapsto i_z \omega \in \Lambda^{q-1}(A),
 \end{aligned}$$

given by

$$(4.3) \quad i_z \omega(z_2, \dots, z_q) = \omega(z, z_2, \dots, z_q),$$

for any $z_2, \dots, z_q \in A$, is called the interior product associated to the element z . Moreover, for any $f \in \mathcal{F}$, we define $i_z f = 0$.

Theorem 4.9. If $z \in A$, then for any $\omega \in \Lambda^q(A)$ and $\theta \in \Lambda^r(A)$ we obtain

$$(4.4) \quad i_z(\omega \wedge \theta) = i_z \omega \wedge \theta + (-1)^q \omega \wedge i_z \theta.$$

Proof. Let $z_1, \dots, z_{q+r} \in A$. Then using the above definition we get

$$\begin{aligned}
 i_{z_1}(\omega \wedge \theta)(z_2, \dots, z_{q+r}) &= (\omega \wedge \theta)(z_1, z_2, \dots, z_{q+r}) \\
 &= \sum_{\substack{\sigma(1) < \dots < \sigma(q) \\ \sigma(q+1) < \dots < \sigma(q+r)}} sgn(\sigma) \omega(z_{\sigma(1)}, \dots, z_{\sigma(q)}) \cdot \theta(z_{\sigma(q+1)}, \dots, z_{\sigma(q+r)}) \\
 &= \sum_{\substack{1=\sigma(1) < \sigma(2) < \dots < \sigma(q) \\ \sigma(q+1) < \dots < \sigma(q+r)}} sgn(\sigma) \omega(z_1, z_{\sigma(2)}, \dots, z_{\sigma(q)}) \cdot \theta(z_{\sigma(q+1)}, \dots, z_{\sigma(q+r)}) \\
 &\quad + \sum_{\substack{\sigma(1) < \dots < \sigma(q) \\ 1=\sigma(q+1) < \sigma(q+2) < \dots < \sigma(q+r)}} sgn(\sigma) \omega(z_{\sigma(1)}, \dots, z_{\sigma(q)}) \cdot \theta(z_1, z_{\sigma(q+2)}, \dots, z_{\sigma(q+r)}) \\
 &= \sum_{\substack{\sigma(2) < \dots < \sigma(q) \\ \sigma(q+1) < \dots < \sigma(q+r)}} sgn(\sigma) i_{z_1} \omega(z_{\sigma(2)}, \dots, z_{\sigma(q)}) \cdot \theta(z_{\sigma(q+1)}, \dots, z_{\sigma(q+r)}) \\
 &\quad + \sum_{\substack{\sigma(1) < \dots < \sigma(q) \\ \sigma(q+2) < \dots < \sigma(q+r)}} sgn(\sigma) \omega(z_{\sigma(1)}, \dots, z_{\sigma(q)}) \cdot i_{z_1} \theta(z_{\sigma(q+2)}, \dots, z_{\sigma(q+r)}).
 \end{aligned}$$

In the second sum, we have the permutation

$$\sigma = \begin{pmatrix} 1 & \dots & q & q+1 & q+2 & \dots & q+r \\ \sigma(1) & \dots & \sigma(q) & 1 & \sigma(q+2) & \dots & \sigma(q+r) \end{pmatrix}.$$

We observe that $\sigma = \tau \circ \tau'$, where

$$\tau = \begin{pmatrix} 1 & 2 & \dots & q+1 & q+2 & \dots & q+r \\ 1 & \sigma(1) & \dots & \sigma(q) & \sigma(q+2) & \dots & \sigma(q+r) \end{pmatrix},$$

and

$$\tau' = \begin{pmatrix} 1 & 2 & \cdots & q & q+1 & q+2 & \cdots & q+r \\ 2 & 3 & \cdots & q+1 & 1 & q+2 & \cdots & q+r \end{pmatrix}.$$

Since $\tau(2) < \cdots < \tau(q+1)$ and τ' has q inversions, it results that

$$\text{sgn}(\sigma) = (-1)^q \text{sgn}(\tau).$$

Therefore

$$\begin{aligned} i_{z_1}(\omega \wedge \theta)(z_2, \dots, z_{q+r}) &= (i_{z_1}\omega \wedge \theta)(z_2, \dots, z_{q+r}) \\ &+ (-1)^q \sum_{\substack{\tau(2) < \cdots < \tau(q) \\ \tau(q+2) < \cdots < \tau(q+r)}} \text{sgn}(\tau) \cdot \omega(z_{\tau(2)}, \dots, z_{\tau(q)}) \cdot i_{z_1}\theta(z_{\tau(q+2)}, \dots, z_{\tau(q+r)}) \\ &= (i_{z_1}\omega \wedge \theta)(z_2, \dots, z_{q+r}) + (-1)^q (\omega \wedge i_{z_1}\theta)(z_2, \dots, z_{q+r}), \end{aligned}$$

which completes the proof. \square

Theorem 4.10. *For any $z, v \in A$, we have*

$$(4.5) \quad L_v \circ i_z - i_z \circ L_v = i_{[v,z]_A}.$$

Proof. Let $\omega \in \Lambda^q(A)$. Then

$$\begin{aligned} i_z(L_v\omega)(z_2, \dots, z_q) &= L_v\omega(z, z_2, \dots, z_q) \\ &= \rho(v)(\omega(z, z_2, \dots, z_q)) - \omega([v, z]_A, z_2, \dots, z_q) \\ &\quad - \sum_{i=2}^q \omega((z, z_2, \dots, [v, z_i]_A, \dots, z_q)) \\ &= \rho(v)(i_z\omega(z_2, \dots, z_q)) - \sum_{i=2}^q i_z\omega(z_2, \dots, [v, z_i]_A, \dots, z_q) \\ &\quad - i_{[v,z]_A}(z_2, \dots, z_q) = (L_v(i_z\omega) - i_{[v,z]_A})(z_2, \dots, z_q), \end{aligned}$$

for any $z_2, \dots, z_q \in A$. \square

Definition 4.11. *If $f \in \mathcal{F}$ and $z \in A$, then the exterior differential operator is defined by*

$$(4.6) \quad d^A f(z) = \rho(z)f.$$

Theorem 4.12. *The operator*

$$\begin{array}{ccc} \Lambda^q(A) & \xrightarrow{d^A} & \Lambda^{q+1}(A) \\ \omega & \longmapsto & d^A\omega \end{array},$$

defined by

$$(4.7) \quad \begin{aligned} d^A\omega(z_0, z_1, \dots, z_q) &= \sum_{i=0}^q (-1)^i \rho(z_i)(\omega(z_0, z_1, \dots, \hat{z}_i, \dots, z_q)) \\ &\quad + \sum_{i < j} (-1)^{i+j} \omega([z_i, z_j]_A, z_0, z_1, \dots, \hat{z}_i, \dots, \hat{z}_j, \dots, z_q), \end{aligned}$$

for any $z_0, z_1, \dots, z_q \in A$, is unique with the following property:

$$(4.8) \quad L_z = d^A \circ i_z + i_z \circ d^A, \quad \forall z \in A.$$

Proof. At first we prove that the equation (4.8) holds. We get

$$\begin{aligned}
 (i_{z_0} \circ d^A)\omega(z_1, \dots, z_q) &= d^A\omega(z_0, z_1, \dots, z_q) \\
 &= \sum_{i=0}^q (-1)^i \rho(z_i)(\omega(z_0, z_1, \dots, \hat{z}_i, \dots, z_q)) \\
 &\quad + \sum_{0 \leq i < j} (-1)^{i+j} \omega([z_i, z_j]_A, z_0, z_1, \dots, \hat{z}_i, \dots, \hat{z}_j, \dots, z_q) \\
 &= \rho(z_0)(\omega(z_1, \dots, z_q)) \\
 &\quad + \sum_{i=1}^q (-1)^i \rho(z_i)(\omega(z_0, z_1, \dots, \hat{z}_i, \dots, z_q)) \\
 &\quad + \sum_{i=1}^q (-1)^i \omega([z_0, z_i]_A, z_1, \dots, \hat{z}_i, \dots, z_q) \\
 &\quad + \sum_{1 \leq i < j} (-1)^{i+j} \omega([z_i, z_j]_A, z_0, z_1, \dots, \hat{z}_i, \dots, \hat{z}_j, \dots, z_q),
 \end{aligned}$$

which gives us

$$\begin{aligned}
 (i_{z_0} \circ d^A)\omega(z_1, \dots, z_q) &= \rho(z_0)(\omega(z_1, \dots, z_q)) \\
 &\quad - \sum_{i=1}^q \omega(z_1, \dots, [z_0, z_i]_A, \dots, z_q) \\
 &\quad - \sum_{i=1}^q (-1)^{i-1} \rho(z_i)(i_{z_0}\omega((z_1, \dots, \hat{z}_i, \dots, z_q))) \\
 &\quad - \sum_{1 \leq i < j} (-1)^{i+j-2} i_{z_0}\omega([z_i, z_j]_A, z_1, \dots, \hat{z}_i, \dots, \hat{z}_j, \dots, z_q) \\
 &= (L_{z_0} - d^A \circ i_{z_0})\omega(z_1, \dots, z_q),
 \end{aligned}$$

for any $z_0, z_1, \dots, z_q \in A$. Thus (4.8) holds. Now, we verify the uniqueness of the operator d^A . Let d'^A be an another exterior differentiation operator satisfying the property (4.8). We consider the set

$$S = \{q \in \mathbb{N} | d^A\omega = d'^A\omega, \forall \omega \in \Lambda^q((F, \nu, N))A\},$$

and we let $z \in (A, [,]_A, \rho)$. We observe that (4.8) is equivalent with

$$(4.9) \quad i_z \circ (d^A - d'^A) + (d^A - d'^A) \circ i_z = 0.$$

Since $i_z f = 0$, for any $f \in \mathcal{F}$, it results that

$$(d^A - d'^A)(f)(z) = 0, \forall f \in \mathcal{F}.$$

Therefore, we obtain $0 \in S$. We now prove that if $q \in S$ then $q + 1 \in S$. Let $\omega \in \Lambda^{p+1}(A)$. Since $i_z \omega \in \Lambda^q(A)$, using the equality (4.9), it results that

$$i_z \circ (d^A - d'^A)\omega = 0,$$

which implies $((d^A - d'^A)\omega)(z_0, z_1, \dots, z_q) = 0$, for any $z_1, \dots, z_q \in A$. Therefore $d^A\omega = d'^A\omega$. Namely $q + 1 \in S$. Thus the Peano's Axiom implies that $S = \mathbb{N}$. Therefore, the uniqueness is verified. \square

The operator given by Theorem 4.12 will be called *the exterior differentiation operator for the exterior differential algebra of the generalized Lie \mathcal{F} -algebra $(A, [,]_A, \rho)$* . Using (4.7) we deduce that if $\omega = \omega_{\alpha_1 \dots \alpha_q} t^{\alpha_1} \wedge \dots \wedge t^{\alpha_q} \in \Lambda^q(A)$, then

$$\begin{aligned} d^A \omega(t_{\alpha_0}, t_{\alpha_1}, \dots, t_{\alpha_q}) &= \sum_{i=0}^q (-1)^i \rho_{\alpha_i}^k \partial_k (\omega_{\alpha_0, \dots, \widehat{\alpha_i} \dots \alpha_q}) \\ &\quad + \sum_{i < j} (-1)^{i+j} L_{\alpha_i \alpha_j}^\alpha \cdot \omega_{\alpha, \alpha_0, \dots, \widehat{\alpha_i}, \dots, \widehat{\alpha_j}, \dots, \alpha_q}. \end{aligned}$$

Therefore, $d^A \omega$ has the following locally expression:

$$\begin{aligned} d^A \omega &= \left(\sum_{i=0}^q (-1)^i \rho_{\alpha_i}^k \partial_k (\omega_{\alpha_0, \dots, \widehat{\alpha_i} \dots \alpha_q}) \right. \\ &\quad \left. + \sum_{i < j} (-1)^{i+j} L_{\alpha_i \alpha_j}^\alpha \cdot \omega_{\alpha, \alpha_0, \dots, \widehat{\alpha_i}, \dots, \widehat{\alpha_j}, \dots, \alpha_q} \right) t^{\alpha_0} \wedge t^{\alpha_1} \wedge \dots \wedge t^{\alpha_q}. \end{aligned}$$

Theorem 4.13. *The exterior differentiation operator d^A has the following properties:*

$$(i) \quad d^A(\omega \wedge \theta) = d^A \omega \wedge \theta + (-1)^q \omega \wedge d^A \theta, \quad \forall \omega \in \Lambda^q(A), \quad \forall \theta \in \Lambda^r(A),$$

$$(ii) \quad L_z \circ d^A = d^A \circ L_z, \quad \forall z \in A,$$

$$(iii) \quad d^A \circ d^A = 0.$$

Proof. (i) Let

$$S = \{q \in \mathbb{N} \mid d^A(\omega \wedge \theta) = d^A \omega \wedge \theta + (-1)^q \omega \wedge d^A \theta, \quad \forall \omega \in \Lambda^q(A)\}.$$

Since

$$\begin{aligned} d^A(f \wedge \omega)(z, v) &= d^A(f \cdot \omega)(z, v) \\ &= \rho(z)(f \omega(v)) - \rho(v)(f \omega(z)) - f \omega([z, v]_A) \\ &= \rho(z)(f) \cdot \omega(v) + f \cdot \rho(z)(\omega(v)) \\ &\quad - \rho(v)(f) \cdot \omega(z) - f \cdot \rho(v)(\omega(z)) - f \omega([z, v]_A) \\ &= d^A f(z) \cdot \omega(v) - d^A f(v) \cdot \omega(z) + f \cdot d^A \omega(z, v) \\ &= (d^A f \wedge \omega)(z, v) + (-1)^0 f \cdot d^A \omega(z, v) \\ &= (d^A f \wedge \omega)(z, v) + (-1)^0 (f \wedge d^A \omega)(z, v), \quad \forall z, v \in A, \end{aligned}$$

then we deduce $0 \in S$. We now prove that if $q \in S$, then $q + 1 \in S$. Without loss of generality, we consider that $\theta \in \Lambda^r(A)$. Then we get

$$\begin{aligned}
 d^A(\omega \wedge \theta)(z_0, z_1, \dots, z_{q+r}) &= i_{z_0} \circ d^A(\omega \wedge \theta)(z_1, \dots, z_{q+r}) \\
 &= L_{z_0}(\omega \wedge \theta)(z_1, \dots, z_{q+r}) - d^A \circ i_{z_0}(\omega \wedge \theta)(z_1, \dots, z_{q+r}) \\
 &= (L_{z_0}\omega \wedge \theta + \omega \wedge L_{z_0}\theta)(z_1, \dots, z_{q+r}) \\
 &\quad - [d^A \circ (i_{z_0}\omega \wedge \theta + (-1)^q \omega \wedge i_{z_0}\theta)](z_1, \dots, z_{q+r}) \\
 &= (L_{z_0}\omega \wedge \theta + \omega \wedge L_{z_0}\theta - (d^A \circ i_{z_0}\omega) \wedge \theta)(z_1, \dots, z_{q+r}) \\
 &\quad - ((-1)^{q-1} i_{z_0}\omega \wedge d^A\theta + (-1)^q d^A\omega \wedge i_{z_0}\theta)(z_1, \dots, z_{q+r}) \\
 &\quad - (-1)^{2q} \omega \wedge d^A \circ i_{z_0}\theta(z_1, \dots, z_{q+r}) \\
 &= ((L_{z_0}\omega - d^A \circ i_{z_0}\omega) \wedge \theta)(z_1, \dots, z_{q+r}) \\
 &\quad + \omega \wedge (L_{z_0}\theta - d^A \circ i_{z_0}\theta)(z_1, \dots, z_{q+r}) \\
 &\quad + ((-1)^q i_{z_0}\omega \wedge d^A\theta - (-1)^q d^A\omega \wedge i_{z_0}\theta)(z_1, \dots, z_{q+r}) \\
 &= [((i_{z_0} \circ d^A)\omega) \wedge \theta + (-1)^{q+1} d^A\omega \wedge i_{z_0}\theta](z_1, \dots, z_{q+r}) \\
 &\quad + [\omega \wedge ((i_{z_0} \circ d^A)\theta) + (-1)^q i_{z_0}\omega \wedge d^A\theta](z_1, \dots, z_{q+r}) \\
 &= [i_{z_0}(d^A\omega \wedge \theta) + (-1)^q i_{z_0}(\omega \wedge d^A\theta)](z_1, \dots, z_{q+r}) \\
 &= [d^A\omega \wedge \theta + (-1)^q \omega \wedge d^A\theta](z_1, \dots, z_{q+r}),
 \end{aligned}$$

for any $z_0, z_1, \dots, z_{q+r} \in A$, which gives us $q + 1 \in S$. Thus using the Peano's Axiom we deduce $S = \mathbb{N}$.

(ii) Let $z \in A$ and we consider

$$S = \{q \in \mathbb{N} | (L_z \circ d^A)\omega = (d^A \circ L_z)\omega, \forall \omega \in \Lambda^q(A)\}.$$

If $f \in \mathcal{F}$, then we get

$$\begin{aligned}
 (d^A \circ L_z)f(v) &= i_v \circ (d^A \circ L_z)f = (i_v \circ d^A) \circ L_z f \\
 &= (L_v \circ L_z)f - ((d^A \circ i_v) \circ L_z)f \\
 &= (L_v \circ L_z)f - L_{[z,v]_A}f + d^A \circ i_{[z,v]_A}f - d^A \circ L_z(i_v f) \\
 &= (L_v \circ L_z)f - L_{[z,v]_A}f + d^A \circ i_{[z,v]_A}f - 0 \\
 &= (L_v \circ L_z)f - L_{[z,v]_A}f + d^A \circ i_{[z,v]_A}f - L_z \circ d^A(i_v f) \\
 &= (L_z \circ i_v)(d^A f) - L_{[z,v]_A}f + d^A \circ i_{[z,v]_A}f \\
 &= (i_v \circ L_z)(d^A f) + L_{[z,v]_A}f - L_{[z,v]_A}f \\
 &= i_v \circ (L_z \circ d^A)f = (L_z \circ d^A)f(v), \forall v \in A,
 \end{aligned}$$

which results that $0 \in S$. We now show that if $q \in S$, then $q + 1 \in S$. Let $\omega \in \Lambda^q(A)$. Then

$$\begin{aligned}
 (d^A \circ L_z)\omega(z_0, z_1, \dots, z_q) &= i_{z_0} \circ (d^A \circ L_z)\omega(z_1, \dots, z_q) \\
 &= (i_{z_0} \circ d^A) \circ L_z \omega(z_1, \dots, z_q) \\
 &= [(L_{z_0} \circ L_z)\omega - ((d^A \circ i_{z_0}) \circ L_z)\omega](z_1, \dots, z_q) \\
 &= [(L_{z_0} \circ L_z)\omega - L_{[z,z_0]_A}\omega](z_1, \dots, z_q) \\
 &\quad + [d^A \circ i_{[z,z_0]_A}\omega - d^A \circ L_z(i_{z_0}\omega)](z_1, \dots, z_q).
 \end{aligned}$$

Using (ii) in the above equation we obtain

$$\begin{aligned}
(d^A \circ L_z)\omega(z_0, z_1, \dots, z_q) &= [(L_{z_0} \circ L_z)\omega - L_{[z, z_0]_A}\omega](z_1, \dots, z_q) \\
&+ [d^A \circ i_{[z, z_0]_A}\omega - L_z \circ d^A(i_{z_0}\omega)](z_1, \dots, z_q) \\
&= [(L_z \circ i_{z_0})(d^A\omega) - L_{[z, z_0]_A}\omega + d^F \circ i_{[z, z_0]_A}\omega](z_1, \dots, z_q) \\
&= [(i_{z_0} \circ L_z)(d^A\omega) + L_{[z, z_0]_A}\omega - L_{[z, z_0]_A}\omega](z_1, \dots, z_q) \\
&= i_{z_0} \circ (L_z \circ d^A)\omega(z_1, \dots, z_q) \\
&= (L_z \circ d^A)\omega(z_0, z_1, \dots, z_q), \quad \forall z_0, z_1, \dots, z_q \in A,
\end{aligned}$$

which implies $q + 1 \in S$. Using the Peano's Axiom we result that $S = \mathbb{N}$.

(iii) It is remarked that

$$\begin{aligned}
i_z \circ (d^A \circ d^A) &= (i_z \circ d^A) \circ d^A = L_z \circ d^A - (d^A \circ i_z) \circ d^A \\
&= L_z \circ d^A - d^A \circ L_z + d^A \circ (d^A \circ i_z) = (d^A \circ d^A) \circ i_z,
\end{aligned}$$

for any $z \in A$. Now, let $\omega \in \Lambda^q(A)$. Then we get

$$\begin{aligned}
(d^A \circ d^A)\omega(z_1, \dots, z_{q+2}) &= i_{z_{q+2}} \circ \dots \circ i_{z_1} \circ (d^A \circ d^A)\omega \\
&= i_{z_{q+2}} \circ (d^A \circ d^A) \circ i_{z_{q+1}}(\omega(z_1, \dots, z_q)) \\
&= i_{z_{q+2}} \circ (d^A \circ d^A)(0) = 0, \quad \forall z_1, \dots, z_{q+2} \in A.
\end{aligned}$$

□

Theorem 4.14. *If d^A is the exterior differentiation operator for the exterior differential \mathcal{F} -algebra $(\Lambda(A), \wedge)$, then we obtain the structure equations of Maurer-Cartan type*

$$(4.10) \quad d^A t^\alpha = -\frac{1}{2} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma,$$

where $\{t^\alpha\}$ is the coframe of the Lie algebra $(A, [,]_A)$.

Proof. Without restriction of generality, we admit that the set of indices of the base of A is ordered. Let α be arbitrary. Since

$$d^A t^\alpha(t_\beta, t_\gamma) = -L_{\beta\gamma}^\alpha, \quad \forall \beta, \gamma,$$

then

$$(4.11) \quad d^A t^\alpha = -\sum_{\beta < \gamma} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma.$$

Since $L_{\beta\gamma}^\alpha = -L_{\gamma\beta}^\alpha$ and $t^\beta \wedge t^\gamma = -t^\gamma \wedge t^\beta$, it results that

$$(4.12) \quad \sum_{\beta < \gamma} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma = \frac{1}{2} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma.$$

(4.11) and (4.12) imply (4.10). □

Equation (4.10) will be called *the structure equations of Maurer-Cartan type associated to the generalized Lie \mathcal{F} -algebra $(A, [,]_A, \rho)$* .

Corollary 4.15. *If d^F is the exterior differentiation operator for the exterior differential $\mathcal{F}(N)$ -algebra $(\Lambda(F, \nu, N), \wedge)$, then locally we obtain the structure equations of Maurer-Cartan type*

$$(C_1) \quad d^F t^\alpha = -\frac{1}{2} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma, \quad \alpha \in \overline{1, p},$$

and

$$(C_2) \quad d^F \varkappa^{\tilde{i}} = \theta_{\alpha}^{\tilde{i}} t^{\alpha}, \quad \tilde{i} \in \overline{1, n},$$

where $\{t^{\alpha}, \alpha \in \overline{1, p}\}$ is the coframe of the vector bundle (F, ν, N) .

This equations will be called *the structure equations of Maurer-Cartan type associated to the generalized Lie algebroid* $((F, \nu, N), [,]_{F,h}, (\rho, \eta))$. In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, the structure equations of Maurer-Cartan type become

$$(C'_1) \quad d^F t^{\alpha} = -\frac{1}{2} L_{\beta\gamma}^{\alpha} t^{\beta} \wedge t^{\gamma}, \quad \alpha \in \overline{1, p},$$

and

$$(C'_2) \quad d^F x^i = \rho_{\alpha}^i t^{\alpha}, \quad i \in \overline{1, m}.$$

Also, in the particular case of standard Lie algebroid, $\rho = Id_{TM}$, the structure equations of Maurer-Cartan type become

$$(C''_1) \quad d^{TM} dx^i = 0, \quad i \in \overline{1, m},$$

and

$$(C''_2) \quad d^{TM} x^i = dx^i, \quad i \in \overline{1, m}.$$

Definition 4.16. For any generalized Lie \mathcal{F} -algebras morphism φ from $(A, [,]_A, \rho)$ to $(A', [,]_{A'}, \rho')$ we define the application

$$\begin{array}{ccc} \Lambda^q(A') & \xrightarrow{\varphi^*} & \Lambda^q(A) \\ \omega' & \mapsto & \varphi^* \omega' \end{array},$$

where

$$(\varphi^* \omega')(z_1, \dots, z_q) = \omega'(\varphi(z_1), \dots, \varphi(z_q)),$$

for any $z_1, \dots, z_q \in A$.

Theorem 4.17. If φ is a generalized Lie \mathcal{F} -algebras morphism from $(A, [,]_A, \rho)$ to $(A', [,]_{A'}, \rho')$, then

$$(i) \quad \varphi^*(\omega' \wedge \theta') = \varphi^* \omega' \wedge \varphi^* \theta', \quad (ii) \quad i_z(\varphi^* \omega') = \varphi^*(i_{\varphi(z)} \omega'), \quad (iii) \quad \varphi^* \circ d^{A'} = d^A \circ \varphi^*,$$

where $z \in A$, $\omega' \in \Lambda^q(A')$ and $\theta' \in \Lambda^r(A')$.

Proof. Let $\omega' \in \Lambda^q(A')$ and $\theta' \in \Lambda^r(A')$. Then we get

$$\begin{aligned} \varphi^*(\omega' \wedge \theta')(z_1, \dots, z_{q+r}) &= (\omega' \wedge \theta')(\varphi(z_1), \dots, \varphi(z_{q+r})) \\ &= \frac{1}{(q+r)!} \sum_{\sigma \in \Sigma_{q+r}} \text{sgn}(\sigma) \cdot \omega'(\varphi(z_1), \dots, \varphi(z_q)) \cdot \theta'(\varphi(z_{q+1}), \dots, \varphi(z_{q+r})) \\ &= \frac{1}{(q+r)!} \sum_{\sigma \in \Sigma_{q+r}} \text{sgn}(\sigma) \cdot \varphi^* \omega'(z_1, \dots, z_q) \varphi^* \theta'(z_{q+1}, \dots, z_{q+r}) \\ &= (\varphi^* \omega' \wedge \varphi^* \theta')(z_1, \dots, z_{q+r}), \end{aligned}$$

which results (i).

Let $z \in A$ and $\omega' \in \Lambda^q(A')$. Then we obtain

$$\begin{aligned} i_z(\varphi^*\omega')(z_2, \dots, z_q) &= \omega'(\varphi(z), \varphi(z_2), \dots, \varphi(z_q)) = i_{\varphi(z)}\omega'(\varphi(z_2), \dots, \varphi(z_q)) \\ &= \varphi^*(i_{\varphi(z)}\omega')(z_2, \dots, z_q), \end{aligned}$$

for any $z_2, \dots, z_q \in A$. Thus (ii) holds.

Let $\omega' \in \Lambda^q(A')$ and $z_0, \dots, z_q \in A$. Then we deduce

$$\begin{aligned} (\varphi^*d^{A'}\omega')(z_0, \dots, z_q) &= (d^{A'}\omega')(\varphi(z_0), \dots, \varphi(z_q)) \\ &= \sum_{i=0}^q (-1)^i \rho'(\varphi(z_i)) \cdot \omega'((\varphi(z_0), \varphi(z_1), \dots, \widehat{\varphi(z_i)}, \dots, \varphi(z_q))) \\ &\quad + \sum_{0 \leq i < j} (-1)^{i+j} \cdot \omega'(\varphi([z_i, z_j]_A), \varphi(z_0), \varphi(z_1), \dots, \widehat{\varphi(z_i)}, \dots, \widehat{\varphi(z_j)}, \dots, \varphi(z_q)), \end{aligned}$$

and

$$\begin{aligned} d^A(\varphi^*\omega')(z_0, \dots, z_q) &= \sum_{i=0}^q (-1)^i \rho(z_i) \cdot (\varphi^*\omega')(z_0, \dots, \widehat{z_i}, \dots, z_q) \\ &\quad + \sum_{0 \leq i < j} (-1)^{i+j} \cdot (\varphi^*\omega')([z_i, z_j]_A, z_0, \dots, \widehat{z_i}, \dots, \widehat{z_j}, \dots, z_q) \\ &= \sum_{i=0}^q (-1)^i \rho(z_i) \cdot \omega'(\varphi(z_0), \dots, \widehat{\varphi(z_i)}, \dots, \varphi(z_q)) \\ &\quad + \sum_{0 \leq i < j} (-1)^{i+j} \cdot \omega'(\varphi([z_i, z_j]_A), \varphi(z_0), \varphi(z_1), \dots, \widehat{\varphi(z_i)}, \dots, \widehat{\varphi(z_j)}, \dots, \varphi(z_q)). \end{aligned}$$

Two above equations imply (iii). \square

Let $\omega \in \Lambda^q(A)$. If $d^A\omega = 0$, then we say that ω is *closed exterior q -form*. Also, if there exists $\eta \in \Lambda^{q-1}(A)$ such that $\omega = d^A\eta$, then we say that ω is *exact exterior q -form*. For any $q \in \overline{1, n}$ we denote by $\mathcal{Z}^q(A)$ and $\mathcal{B}^q(A)$, the set of closed exterior q -forms and the set of exact exterior q -forms, respectively. Indeed we have

$$\begin{aligned} \mathcal{Z}^q(A) &= \{\omega \in \Lambda^q(A) | d^A\omega = 0\}, \\ \mathcal{B}^q(A) &= \{\omega \in \Lambda^q(A) | \exists \eta \in \Lambda^{q-1}(A) \text{ s.t. } d^A\eta = \omega\}. \end{aligned}$$

Definition 4.18. If $((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta))$ is a generalized Lie algebroid, then the exterior differential calculus of its generalized Lie $\mathcal{F}(N)$ -algebra is called *exterior differential calculus of the generalized Lie algebroid $((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta))$* .

5. INTERIOR AND EXTERIOR ALGEBRAIC/DIFFERENTIAL SYSTEMS

Let $(A, [\cdot, \cdot]_A, \rho)$ be a generalized Lie \mathcal{F} -algebra such that $\dim_{\mathcal{F}} A = p$ and $((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta))$ be a generalized Lie algebroid.

Definition 5.1. Any \mathcal{F} -submodule E of the \mathcal{F} -module A will be called *interior algebraic system (IAS) of the generalized Lie \mathcal{F} -algebra $(A, [\cdot, \cdot]_A, \rho)$* .

Remark 5.2. If E is an IAS of the generalized Lie \mathcal{F} -algebra $(A, [\cdot, \cdot]_A, \rho)$, then we obtain an \mathcal{F} -submodule E^0 of the \mathcal{F} -module A^* such that

$$E^0 \stackrel{\text{put}}{=} \{\Omega \in A^* | \Omega(u) = 0, \forall u \in E\}.$$

The \mathcal{F} -submodule E^0 will be called *the annihilator \mathcal{F} -submodule of the interior algebraic system E* .

Definition 5.3. Any vector subbundle (E, π, M) of the pull-back vector bundle $(h^*F, h^*\nu, M)$ will be called an *interior differential system (IDS) of the generalized Lie algebroid*

$$((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta)).$$

In particular, if $h = Id_N = \eta$, then we obtain the definition of *IDS* of a Lie algebroid (see [5]).

Remark 5.4. If (E, π, M) is an *IDS* of the generalized Lie algebroid

$$((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta)),$$

then we obtain a vector subbundle (E^0, π^0, M) of the vector bundle $(h^*F, h^*\nu, M)$ such that

$$\Gamma(E^0, \pi^0, M) \stackrel{put}{=} \left\{ \Omega \in \Gamma(h^*F, h^*\nu, M) \mid \Omega(S) = 0, \forall S \in \Gamma(E, \pi, M) \right\}.$$

The vector subbundle (E^0, π^0, M) will be called *the annihilator vector subbundle of the interior differential system (E, π, M)* .

Easily we can deduce the following proposition:

Proposition 5.5. If E is an *IAS* of the generalized Lie \mathcal{F} -algebra $(A, [\cdot, \cdot]_A, \rho)$ such that $E = \langle s_1, \dots, s_r \rangle$, then there exist $\theta^{r+1}, \dots, \theta^p \in A^*$ linearly independent such that $E^0 = \langle \theta^{r+1}, \dots, \theta^p \rangle$.

Proposition 5.6. If (E, π, M) is an *IDS* of the generalized Lie algebroid

$$((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta)),$$

such that for any $(U, \xi_U) \in [\mathcal{A}_M]$ we have $\Gamma(E|_U, \pi, M) = \langle S_1, \dots, S_r \rangle$, then there exist $\Theta^{r+1}, \dots, \Theta^p \in \Gamma(h^*F|_U, h^*\nu, U)$ linearly independent such that $\Gamma(E|_U^0, \pi^0, U) = \langle \Theta^{r+1}, \dots, \Theta^p \rangle$.

The interior algebraic system E of the generalized Lie \mathcal{F} -algebra $(A, [\cdot, \cdot]_A, \rho)$ is called *involutive* if $[u, v]_A \in E$, for any $u, v \in E$. Also, the interior differential system (E, π, M) of the generalized Lie algebroid

$$((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta)),$$

is called *involutive* if $[S, T]_{h^*F} \in \Gamma(E, \pi, M)$, for any $S, T \in \Gamma(E, \pi, M)$. Using these definitions, we can deduce the following properties:

Proposition 5.7. If E is an *IAS* of the generalized Lie \mathcal{F} -algebra

$(A, [\cdot, \cdot]_A, \rho)$ and $\{s_1, \dots, s_r\}$ is a basis for the \mathcal{F} -submodule E then E is involutive if and only if $[s_a, s_b]_A \in E$, for any $a, b \in \overline{1, r}$.

Proposition 5.8. Let (E, π, M) be an *IDS* of the generalized Lie algebroid

$$((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta)).$$

If for any $(U, \xi_U) \in [\mathcal{A}_M]$ there exists a basis $\{S_1, \dots, S_r\}$ for the $\mathcal{F}(M)|_U$ -submodule $\Gamma(E|_U, \pi, U)$, then (E, π, M) is involutive if and only if

$$[S_a, S_b]_{h^*F|_U} \in \Gamma(E|_U, \pi, U),$$

for for any $a, b \in \overline{1, r}$.

Theorem 5.9. (*Frobenius type*) Let E be an interior algebraic system of the generalized Lie \mathcal{F} -algebra $(A, [\cdot, \cdot]_A, \rho)$. If $\{\theta^{r+1}, \dots, \theta^p\}$ is a basis for the annihilator submodule E^0 , then E is involutive if and only if there exist

$$\omega_\beta^\alpha \in \Lambda^1(A), \quad \alpha, \beta \in \overline{r+1, p},$$

such that

$$(5.1) \quad d^A \theta^\alpha = \sum_{\beta \in \overline{r+1, p}} \omega_\beta^\alpha \wedge \theta^\beta, \quad \alpha \in \overline{r+1, p}.$$

Proof. Let $\{s_1, \dots, s_r\}$ be a basis for the \mathcal{F} -submodule E and we suppose that $s_{r+1}, \dots, s_p \in A$ such that

$$\{s_1, \dots, s_r, s_{r+1}, \dots, s_p\},$$

is a basis for the \mathcal{F} -module A . Also, let $\theta^1, \dots, \theta^r \in A^*$ such that

$$\{\theta^1, \dots, \theta^r, \theta^{r+1}, \dots, \theta^p\},$$

be a basis for the \mathcal{F} -module A^* . For any $a, b \in \overline{1, r}$ and $\alpha, \beta \in \overline{r+1, p}$, we have the equalities:

$$\theta^a(s_b) = \delta_b^a, \quad \theta^a(s_\beta) = 0, \quad \theta^\alpha(s_b) = 0, \quad \theta^\alpha(s_\beta) = \delta_\beta^\alpha.$$

We remark that the set of the 2-forms

$$\{\theta^a \wedge \theta^b, \theta^a \wedge \theta^\beta, \theta^\alpha \wedge \theta^\beta, \quad a, b \in \overline{1, r} \wedge \alpha, \beta \in \overline{r+1, p}\},$$

is a base for the \mathcal{F} -module $\Lambda^2(A)$. Therefore, we have

$$(5.2) \quad d^A \theta^\alpha = \sum_{b < c} A_{bc}^\alpha \theta^b \wedge \theta^c + \sum_{b, \gamma} B_{b\gamma}^\alpha \theta^b \wedge \theta^\gamma + \sum_{\beta < \gamma} C_{\beta\gamma}^\alpha \theta^\beta \wedge \theta^\gamma,$$

where, $A_{bc}^\alpha, B_{b\gamma}^\alpha$ and $C_{\beta\gamma}^\alpha$, $a, b, c \in \overline{1, r} \wedge \alpha, \beta, \gamma \in \overline{r+1, p}$ are components such that $A_{bc}^\alpha = -A_{cb}^\alpha$ and $C_{\beta\gamma}^\alpha = -C_{\gamma\beta}^\alpha$. Using the formula

$$(5.3) \quad d^A \theta^\alpha(s_b, s_c) = \rho(s_b)(\theta^\alpha(s_c)) - \rho(s_c)(\theta^\alpha(s_b)) - \theta^\alpha([s_b, s_c]_A),$$

we obtain

$$(5.4) \quad A_{bc}^\alpha = -\theta^\alpha([s_b, s_c]_A), \quad \forall (b, c \in \overline{1, r} \wedge \alpha \in \overline{r+1, p}).$$

We admit that E is an involutive IAS of the generalized Lie \mathcal{F} -algebra $(A, [\cdot, \cdot]_A, \rho)$. Since $[s_b, s_c]_A \in E$, $\forall b, c \in \overline{1, r}$, then it results that $\theta^\alpha([s_b, s_c]_A) = 0$, where $\alpha \in \overline{r+1, p}$. Therefore $A_{bc}^\alpha = 0$, and consequently

$$\begin{aligned} d^A \theta^\alpha &= \sum_{b, \gamma} B_{b\gamma}^\alpha \theta^b \wedge \theta^\gamma + \frac{1}{2} C_{\beta\gamma}^\alpha \theta^\beta \wedge \theta^\gamma \\ &= (B_{b\gamma}^\alpha \theta^b + \frac{1}{2} C_{\beta\gamma}^\alpha \theta^\beta) \wedge \theta^\gamma, \quad \forall \alpha, \beta, \gamma \in \overline{r+1, p}. \end{aligned}$$

Setting

$$\omega_\gamma^\alpha \stackrel{\text{put}}{=} B_{b\gamma}^\alpha \theta^b + \frac{1}{2} C_{\beta\gamma}^\alpha \theta^\beta \in \Lambda^1(A),$$

in the above equation the necessity condition of assertion proves. Conversely, we admit that there exist $\omega_\beta^\alpha \in \Lambda^1(A)$, $\alpha, \beta \in \overline{r+1, p}$, such that

$$(5.5) \quad d^A \theta^\alpha = \sum_{\beta \in \overline{r+1, p}} \omega_\beta^\alpha \wedge \theta^\beta.$$

Using the affirmations (5.2), (5.3) and (5.5) we derive that

$$A_{bc}^\alpha = 0, \quad \forall b, c \in \overline{1, r}, \quad \forall \alpha \in \overline{r+1, p}.$$

Thus using (5.4), we obtain

$$\theta^\alpha([s_b, s_c]_A) = 0, \quad \forall (b, c \in \overline{1, r} \wedge \alpha \in \overline{r+1, p}),$$

which gives us

$$[s_b, s_c]_A \in E, \quad \forall b, c \in \overline{1, r}.$$

Therefore from previous proposition we deduce that E is involutive. \square

Corollary 5.10. *(of Frobenius type) Let (E, π, M) be an IDS of the generalized Lie algebroid $((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta))$. If for any $(U, \xi_U) \in [\mathcal{A}_M]$, there exists the basis $\{\Theta^{r+1}, \dots, \Theta^p\}$ for the $\mathcal{F}(M)|_U$ -submodule $\Gamma(E|_U^0, \pi^0, U)$, then (E, π, M) is involutive if and only if there exist*

$$\Omega_\beta^\alpha \in \Lambda^1(h^*F|_U, h^*\nu, U), \quad \alpha, \beta \in \overline{r+1, p}$$

such that

$$d^{h^*F}\Theta^\alpha = \sum_{\beta \in \overline{r+1, p}} \Omega_\beta^\alpha \wedge \Theta^\beta, \quad \alpha \in \overline{r+1, p}.$$

Definition 5.11. *An \mathcal{F} -submodule \mathcal{I} of the Lie \mathcal{F} -algebra $(A, [\cdot, \cdot]_A)$ such that $[u, v]_A \in \mathcal{I}$, for any $u \in A$ and $v \in \mathcal{I}$, is called the ideal of the Lie \mathcal{F} -algebra $(A, [\cdot, \cdot]_A)$.*

Remark 5.12. *If E is a \mathcal{F} -submodule of the Lie \mathcal{F} -algebra $(A, [\cdot, \cdot]_A)$ and*

$$\mathcal{I}(E) \stackrel{put}{=} \bigcap_{\substack{\mathcal{I} = ideal \\ \mathcal{I} \supseteq E}} \mathcal{I},$$

then $\mathcal{I}(E)$ is an ideal of the Lie \mathcal{F} -algebra $(A, [\cdot, \cdot]_A)$ which is called the ideal generated by the \mathcal{F} -submodule E .

Definition 5.13. *An ideal \mathcal{I} of the exterior algebra of the generalized Lie \mathcal{F} -algebra $(A, [\cdot, \cdot]_A, \rho)$ closed under differentiation operator d^A , namely $d^A\mathcal{I} \subseteq \mathcal{I}$, is called a differential ideal of the generalized Lie \mathcal{F} -algebra $(A, [\cdot, \cdot]_A, \rho)$.*

Let \mathcal{I} be a differential ideal of the generalized Lie \mathcal{F} -algebra $(A, [\cdot, \cdot]_A, \rho)$. If there exists an *interior algebraic system* E such that for all $k \in \mathbb{N}^*$ and $\omega \in \mathcal{I} \cap \Lambda^k(A)$ we have $\omega(u_1, \dots, u_k) = 0$, for any $u_1, \dots, u_k \in E$, then we will say that \mathcal{I} is an *exterior algebraic system of the generalized Lie \mathcal{F} -algebra $(A, [\cdot, \cdot]_A, \rho)$.*

Definition 5.14. *Any ideal \mathcal{I} of the exterior differential algebra of the pull-back Lie algebroid*

$$((h^*F, h^*\nu, M), [\cdot, \cdot]_{h^*F}, (\overset{h^*F}{\rho}, Id_M))$$

*closed under differentiation operator d^{h^*F} , namely $d^{h^*F}\mathcal{I} \subseteq \mathcal{I}$, will be called a differential ideal of the generalized Lie algebroid $((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta))$.*

Let \mathcal{I} be a differential ideal of the generalized Lie algebroid $((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta))$. If there exists an IDS (E, π, M) such that for all $k \in \mathbb{N}^*$ and $\omega \in \mathcal{I} \cap \Lambda^k(h^*F, h^*\nu, M)$ we have $\omega(u_1, \dots, u_k) = 0$, for any $u_1, \dots, u_k \in \Gamma(E, \pi, M)$, then we will say that \mathcal{I} is an *exterior differential system of the generalized Lie algebroid*

$$((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta)).$$

In particular, if $h = Id_N = \eta$, then we obtain the definition of the EDS of a Lie algebroid (see[5]).

Theorem 5.15. (*Cartan type*) *The interior algebraic system E of the generalized Lie \mathcal{F} -algebra $(A, [,]_A, \rho)$ is involutive, if and only if the ideal generated by the \mathcal{F} -submodule E^0 is an exterior algebraic system of the same generalized Lie \mathcal{F} -algebra $(A, [,]_A, \rho)$.*

Proof. Let E be an involutive interior algebraic system of the generalized Lie \mathcal{F} -algebra $(A, [,]_A, \rho)$ and let $\{\theta^{r+1}, \dots, \theta^p\}$ be a basis for the \mathcal{F} -submodule E^0 . We know that

$$\mathcal{I}(E^0) = \cup_{q \in \mathbb{N}} \{\omega_\alpha \wedge \theta^\alpha, \{\omega_{r+1}, \dots, \omega_p\} \subset \Lambda^q(A)\}.$$

Let $q \in \mathbb{N}$ and $\{\omega_{r+1}, \dots, \omega_p\} \subset \Lambda^q(A)$. Using Theorems 4.13 and 5.9 we obtain

$$\begin{aligned} d^A(\omega_\alpha \wedge \theta^\alpha) &= d^A\omega_\alpha \wedge \theta^\alpha + (-1)\omega_\beta^{q+1} \wedge d^A\theta^\beta \\ &= (d^A\omega_\alpha + (-1)\omega_\beta^{q+1} \wedge \omega_\alpha^\beta) \wedge \theta^\alpha. \end{aligned}$$

Since

$$d^A\omega_\alpha + (-1)\omega_\beta^{q+1} \wedge \omega_\alpha^\beta \in \Lambda^{q+2}(A),$$

then we get $d^A(\omega_\beta \wedge \theta^\beta) \in \mathcal{I}(E^0)$, and consequently $d^A\mathcal{I}(E^0) \subseteq \mathcal{I}(E^0)$.

Conversely, let E be an interior algebraic system of the generalized Lie \mathcal{F} -algebra $(A, [,]_A, \rho)$ such that the \mathcal{F} -submodule $\mathcal{I}(E^0)$ is an exterior algebraic system of the generalized Lie \mathcal{F} -algebra $(A, [,]_A, \rho)$. Suppose that $\{\theta^{r+1}, \dots, \theta^p\}$ is a basis for the \mathcal{F} -submodule E^0 . Since $d^A\mathcal{I}(E^0) \subseteq \mathcal{I}(E^0)$, then there exist $\omega_\beta^\alpha \in \Lambda^1(A)$, $\alpha, \beta \in \overline{r+1, p}$, such that

$$d^A\theta^\alpha = \sum_{\beta \in \overline{r+1, p}} \omega_\beta^\alpha \wedge \theta^\beta \in \mathcal{I}(E^0).$$

Using Theorem 5.9, it results that E is an involutive interior algebraic system of the generalized Lie \mathcal{F} -algebra $(A, [,]_A, \rho)$. \square

Corollary 5.16. *The interior differential system (E, π, M) of the generalized Lie algebroid*

$$((F, \nu, N), [,]_{F,h}, (\rho, \eta)),$$

is involutive, if and only if for any $(U, \xi_U) \in [\mathcal{A}_M]$ the ideal generated by the $\mathcal{F}(M)_{|U}$ -submodule $\Gamma(E^0_{|U}, \pi^0, U)$ is an EDS of the same generalized Lie algebroid.

6. NEW DIRECTIONS BY RESEARCH

We know that a generalized Lie \mathcal{F} -algebras morphism from $(A, [,]_A, \rho)$ to $(A', [,]_{A'}, \rho')$ is a Lie \mathcal{F} -algebras morphism φ from $(A, [,]_A)$ to $(A', [,]_{A'})$ such that $\rho' \circ \varphi = \rho$.

Definition 6.1. *An algebraic symplectic \mathcal{F} -space is a pair $((A, [,]_A, \rho), \omega)$ consisting by a generalized Lie \mathcal{F} -algebra $(A, [,]_A, \rho)$ and a nondegenerate closed 2-form $\omega \in \Lambda^2(A)$.*

If $((A', [,]_{A'}, \rho'), \omega')$ is an another generalized symplectic \mathcal{F} -space, then we can define the set of morphisms from $((A, [,]_A, \rho), \omega)$ to $((A', [,]_{A'}, \rho'), \omega')$ as being the set of generalized Lie \mathcal{F} -algebras morphisms φ such that $\varphi^*(\omega') = \omega$. So, we can discuss about the category of algebraic symplectic \mathcal{F} -spaces as being a subcategory of the category of generalized Lie \mathcal{F} -algebras. An algebraic study of objects of this category is a new direction by research.

It is known that the set of morphisms from $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))$ to $((F', \nu', N'), [\cdot]_{F',h'}, (\rho', \eta'))$ is the set of vector bundles morphisms (φ, φ_0) from (F, ν, N) to (F', ν', N') such that φ_0 is diffeomorphism and the modules morphism $\Gamma(\varphi, \varphi_0)$ is a Lie $\mathcal{F}(N)$ -algebras morphism from $(\Gamma(F, \nu, N), [\cdot]_{F,h})$ to $(\Gamma(F', \nu', N'), [\cdot]_{F',h'})$ (see [6]). We can define the *differential symplectic space* as being a pair

$$(((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta)), \omega),$$

consisting of a generalized Lie algebroid $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))$ and a nondegenerate closed 2-form $\omega \in \Lambda^2(F, \nu, N)$. If $((F', \nu', N'), [\cdot]_{F',h'}, (\rho', \eta'))$, ω' is an another differential symplectic space, then we can define the set of morphisms from $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta)), \omega$ to $((F', \nu', N'), [\cdot]_{F',h'}, (\rho', \eta')), \omega'$ as being the set of generalized Lie algebroids morphisms (φ, φ_0) such that

$$(\varphi, \varphi_0)^*(\omega') = \omega.$$

So, we can discuss about the category of differential symplectic spaces as being a subcategory of the category of generalized Lie algebroids. The study of the geometry of objects of this category is an another direction by research.

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